REPARAMETRIZATION INVARIANT OPERATORS IN STRING FIELD THEORY

By

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To all of my teachers throughout the
years, especially my family, who had the
hardest job.

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In this study, the role of the reparametrization group in the field theory of strings is examined. Representations and direct product rules are constructed for both the bosonic and supersymmetric reparametrization algebras. This group theory is applied to free open bosonic string field theory. A catalog is made of the various invariant operators which may be formed. These are found to include a previously unknown symmetric tensor whose trace is the BRST charge. It is found that only certain subsets of the projections of this tensor are closed algebraically. The possible subgroups of the Lorentz group which may be left unbroken by the choice of these subsets are classified. Finally, reparametrization invariance is discussed in the context of the interacting open string theory of Witten. It is argued that in this theory, reparametrization invariance is hidden by a parametrization dependent choice of phases in a set of basis wave functionals. An alternative, completely equivalent, version of this theory is constructed, which does possess manifest reparametrization invariance; however, the interaction in the new formulation has an apparently nonlocal form.

INTRODUCTION

The standard model of particle physics (see ref. 1 for reviews) is in agreement with the results of all high energy physics experiments which have been performed so far. Nevertheless, this model suffers from two shortcomings. Firstly, many details of the model are not predicted by the theory and must be put in "by hand." It is not just the numerical quantities like mass ratios and couplings, but also the pattern of particle families, the gauge group structure, even the fact that spacetime has four dimensions, which the theory makes no attempt to predict. Secondly, the standard model does not include gravity. This does not contradict experiment, because all experiments to date in elementary particle physics have probed physics at energy ranges far lower than those at which the gravitational forces would become significant. However, it does indicate that the theory is not fundamentally complete. It also means that the theory cannot be applied to situations in which both quantum mechanics and gravity are significant, a striking example of which is the early universe.

When one tries to define a quantum field theory containing gravity [2], one quickly finds that the theory is non-renormalizable. This means that the theory at high energies behaves badly enough that its predictive power is lost, at least if one treats the interactions using the standard perturbative methods, which are the only methods currently known which are generally useful in interacting field theory. All of this seems to indicate that some new theory may be needed, one which reduces to the old theory in the appropriate limit.

In contrast to local field theories, there are only a few different string theories [3,4,5]. The simplest is the bosonic string [6,7], which has twenty-six

[8,9,10,11] spacetime coordinates $x^\mu(\sigma)$, where $\mu = 0, 1, ..., 25$ labels the one time and twenty-five space coordinates of the string, and σ is a parameter labeling the individual points along the string. The slightly more complicated superstring [12] theories, of which there are only a few types, include ten coordinates x^μ as above, and anticommuting coordinates which are the supersymmetric partners of the x^μ. There is also a theory called the heterotic string [13], which combines aspects of both the bosonic and supersymmetric theories. This theory has a naturally arising gauge symmetry which contains (among many other things) the gauge groups and structure of the standard model; thus the heterotic string is the one which actually has the possibility of being a workable theory of everything.

In addition to the different categorizations described above, strings have the property of being either open or closed, closed strings being those whose endpoints are constrained to coincide with one another: $x^\mu(0) = x^\mu(2\pi)$. Because of the nature of string interactions, an interacting theory which contains open strings must also contain closed strings. This is because two open strings interact by joining at their endpoints to form one longer open string. Locally, this interaction looks the same as the joining of the two ends of an open string to form a closed string. The end of an open string cannot discriminate between its own other end and the end of another open string without violating the locality of the interactions, thus ruining the quantum consistency of the theory. Of the string theories mentioned above, the heterotic string and most types of superstrings include only closed strings, the other type of superstrings includes both, and the bosonic theory includes closed strings and may or may not include open strings.

In short, there are only a few different string theories, and each has no free parameters. Thus, given enough calculational facility, we could use string theory to predict answers to all experimental questions. If any of these predictions were not correct, string theory would be ruled out as the theory of the universe.

On first examination, the phenomenology of the quantum theory of strings would seem to bear little relation to the experimentally observed world. Heterotic string theory requires ten spacetime dimensions, and predicts a large spectra of particles which are generally not observed. However, various schemes have been proposed by which the ten dimensions of the heterotic string compactify to yield a four-dimensional theory. There seem to be a huge number of such schemes [14,15], each yielding different low-energy phenomenology. Although the dynamics of the string should in principle determine which scheme (if any) is favored, in practice it is not known how to make this determination. One such scheme has been extensively investigated and found to yield realistic results [16]; however, this is only one of thousands or millions of possible schemes.

The spectrum of the closed string is found [17,18] to automatically include a massless spin-two particle, which may be identified as the quantum of the gravitational field. This is of particular interest because string theories are well-behaved in the high-energy regime, unlike other theories of gravity. Thus string theory gives us at best a true theory of everything, and at worst a way of including perturbative gravity into a quantum theory without losing predictive power through nonrenormalizibility. Even if strings turn out not to explain the entire universe, they will certainly show us new directions in which to search for a quantum mechanical theory of gravity.

The standard approach to string theory is to first solve the quantum mechanics of the free string; then interactions are introduced in a way which is manifestly perturbative [19,11,20,21]. Although this is sufficient for many types of calculations, we would expect that a fully nonperturbative starting point would be necessary before we could answer certain questions, such as whether compactification is favored. Such a formulation of the theory is provided by the second-quantized theory of strings, known as string field theory [22,23,24,25,26].

String field theory (for reviews see references 25 and 26) was developed long ago [22] in the light cone gauge [11]. It was not until much later that manifestly covariant field theories of strings were developed. The field theory of open bosonic strings was developed by Witten [24]. This theory was shown to reproduce the correct dual model amplitudes. It is based on a physically appealing picture, as we shall discuss, in which three strings are created from the vacuum at some instant of time. Energy conservation requires one or two of the strings to be created moving back in time. Additionally, the strings are required to overlap, so that half of one string ($0 < \sigma < \pi/2$) is required to overlap with the other half of another string ($\pi/2 < \sigma < \pi$). The generalization of Witten's formalism to open superstrings [27] and closed strings [28] proved to be fraught with difficulties [29,30]; in fact, the question of closed string field theory seems not to be settled [31].

In any string theory one naively expects that observable quantities should be independent of the parametrization of the string. This requirement plays a central role in the first-quantized formalism [5]. However, Witten's string field theory breaks reparametrization invariance by singling out the "middle point," $\sigma = \pi/2$, the location of which clearly depends on a particular parametrization.

As Witten [24] points out, BRST [32,33] invariance is sufficient to guarantee that the negative norm states decouple [34] from the physical states in the theory, so reparametrization invariance is not necessary. Nevertheless, it seems paradoxical that this theory, which completely reproduces the results of the first-quantized theory, disregards a symmetry which was fundamental in the original theory.

In the case of Yang-Mills field theory [35], BRST invariance arises as a remnant of a gauge invariance after the gauge has been fixed. It has been conjectured that Witten's theory may be some gauge-fixed version of a more fundamental theory, one which includes reparametrization invariance, and probably closed strings as well. There have been many attempts [36,37,38] to develop a new version of Witten's interaction which does not lose reparametrization invariance.

This dissertation is organized as follows: Following this introduction, an attempt is made to develop a systematic group theoretical technology of the reparametrization group; for completeness, the supersymmetric extension of the reparametrization group is also discussed. Then the role of the reparametrization group in the free string field theory is examined, in particular the transition from first quantized strings to string fields is discussed. A catalog of operators which are invariant under reparametrizations is presented; some familiar ones are seen, along with a collection of new ones, whose utility is not clear. Finally, we address the question of the apparent lack of reparametrization invariance in the interactions; this is explained as being due to a parametrization-dependent choice of phases in the string field. It is shown how to derive the interactions for any choice of phases, and the interactions are derived for the special case in which the phase convention does not depend on

the parametrization. In this formulation the interactions are shown to be manifestly parametrization invariant, and independent of any choice of a midpoint; however, the interactions appear to be nonlocal in coordinate space.

REPARAMETRIZATION GROUP THEORY

We turn now to a discussion of the mathematical theory associated with the reparametrization symmetry. Generally, the set of symmetry operations which leave a physical system unchanged will necessarily satisfy the four axioms of a group [39]. Namely, the successive action of two symmetry transformations must itself be a symmetry transformation; this way of combining symmetries is associative; there is an identity transformation, which is to do nothing; and any transformation may be reversed so that the net result is no transformation. Thus we may apply the powerful machinery of group theory to study any system with symmetry.

Consider a set of quantities $a_1, a_2, ..., a_N$, which under some symmetry operation T transform to $\overline{a}_i = D(T)_{ij} a_j$, where $D(T)$ is a matrix of numbers, and summation over repeated indices is implied. The matrices D must obey the same product law as the abstract group elements, i.e. $D(T_2 T_1) = D(T_1)D(T_2)$, for consistency. Then the set of matrices D is what is referred to as a representation of the group.

The classification of all possible representations of a given group is a standard problem in group theory. Any group will have a trivial representation in which every element is represented by the identity matrix ($D(T)_{ij} = \delta_{ij}$ for all T), i.e., the quantities a_i are invariant; the transformations have no effect on them at all. One way of generating other representations is to start from a set a_i which are not invariant, and consider all possible products $a_i a_j$. These products may themselves be arranged into representations of the group. Alternately, we may use this procedure to search for some particular representation.

For example, we can use the product laws for representations to determine all possible invariant products which may be formed starting from a given set of quantities. These are necessary and useful in many ways; for example, in constructing an action for a system which must be invariant under a given symmetry group.

The reparametrization group is an example of a Lie group, which means that it has an infinite number of elements, which are labeled by continuous parameters. In such a group, we may consider elements which are infinitesimally close to the identity; these may be expanded as $T(\epsilon_1, \epsilon_2, ...) = 1 + \epsilon_i M_i$, where ϵ_i are infinitesimal parameters. The operators M_i are known as the generators of the Lie algebra of the group. Since the product of two T's must give a certain combination of T's, we find that the commutator of two M's must give a certain combination of M's:

$$[M_i, M_j] = i f_{ijk} M_k \tag{1}$$

The numbers f_{ijk} are known as the structure constants of the algebra. Any matrices which obey (1) provide a representation of the algebra.

Representations of the Reparametrization Algebra

We now apply the above general ideas to the special case of the reparametrization group. Consider the open string whose coordinates are $x_\mu(\sigma)$ where the parameter σ ranges from 0 to π. The parameter has no physical meaning, it is merely a set of numbers which we associate with the points on the string in order to facilitate a mathematical description of it. For a given string there are many ways to attach such numbers, and we would expect the physics of the string not to depend on the particular choice used. Thus under a reparametrization $\sigma \to \overline{\sigma}$ the dynamics should be unchanged. Under such a shift, the

value of the coordinates x_μ at some particular point will not change; however, the functional form of $x^\mu(\sigma)$ will look different because it is written in terms of the new parameter. That is, $x^\mu(\sigma)$ will transform to a new function $\overline{x}^\mu(\sigma)$ such that $\overline{x}^\mu(\overline{\sigma}(\sigma)) = x^\mu(\sigma)$ for all σ. For an infinitesimal reparametrization $\overline{\sigma} = \sigma + \epsilon f(\sigma)$, where ϵ is an infinitesimal parameter,

$$\delta_f(\epsilon)x^\mu(\sigma) \equiv \overline{x}^\mu(\sigma) - x^\mu(\sigma) = -\epsilon f(\sigma)x'(\sigma) \tag{2}$$

where the primes denote derivatives with respect to σ. For open strings, we require $f = 0$ at the string's endpoints, as the location of the endpoints is independent of the parametrization; they are fixed at 0 and π. However, for mathematical purposes the reparametrizations can be formally extended to include any arbitrary function f. For closed strings, f could be any periodic function on the interval 0 to 2π.

From (2) we may calculate the algebra of reparametrizations [23]

$$[\delta_f(\epsilon_1), \delta_g(\epsilon_2)] = \delta_{fg'-f'g}(\epsilon_1\epsilon_2) \tag{3}$$

We are now interested in finding other representations of this algebra. Let us concentrate specifically on representations using a single quantity $q(\sigma)$, whose transformation law will not involve any other quantities. We can narrow down the possibilities by considering a uniform rescaling of the parameter, $\sigma \rightarrow \alpha\sigma$; this should clearly have no effect, therefore each term in the transformation law should contain exactly one derivative with respect to σ. Thus the most general possible transformation law is (suppressing the parameters ϵ) $\delta_f q = -(vfq' + wf'q + zf')$ for some numbers v, w, z. Then we find that (3) is satisfied if and only if $v = 1$:

$$\delta_f q = -(fq' + wf'q + zf') \tag{4}$$

We will refer to such a quantity q as transforming according to the representation (w, z). For the special case $z = 0$, q is said to transform covariantly with weight w. A quantity with $w = 1$ transforms into a total derivative, so its integral over σ is a reparametrization invariant, the contributions from the endpoints vanishing due to the open string boundary condition $f = 0$ at $\sigma = 0, \pi$ (in the case of closed string boundary conditions there are also no such contributions because in this case f must be periodic). An example [23] of a classical invariant which can be constructed from the string coordinates $x^\mu(\sigma)$ is the length of the string

$$l = \int \frac{d\sigma}{\pi} \sqrt{x'^2}, \tag{5}$$

which contains a square root, a harbinger of trouble at the quantum level.

Direct Product Rules

We now examine the transformations of products of two quantities $p(\sigma)$ and $q(\sigma)$. Following the usual group-theoretical treatment, we attempt to identify combinations of the form

$$r(\sigma) \equiv \int d\sigma_1 d\sigma_2 K(\sigma, \sigma_1, \sigma_2) p(\sigma_1) q(\sigma_2)$$

which transform into themselves. This turns out to only be possible in the special case in which p and q transform covariantly. For this case we find two covariant combinations [40,41]:

$$p(\sigma)q(\sigma), \text{ with weight } w_p + w_q \tag{6a}$$

and

$$w_p p q' - w_q p' q, \text{ with weight } w_p + w_q + 1 \tag{6b}$$

where w_p, w_q are the weights of p, q respectively. For $w_p + w_q = 1$, (6 a) will yield an invariant quantity when integrated. However, for $w_p + w_q = 0$, (6 b) is itself a total derivative, so its integral will be a trivial invariant (i.e. made up entirely of endpoint contributions). It would be possible to get a nontrivial invariant by (for example) combining the result of (6 b) with a third quantity according to (6 a), such that the final result had weight one.

The above discussion does not take into account any ordering ambiguities which may occur when the quantities used are operators in the Hilbert space of the first-quantized string. The transformation properties of the product may depend on the ordering of terms within the product, and this must be checked in each case. Generally, the products of interest will be the normal-ordered products. The implementation of the reparametrization group at the quantum level hinges on the existence of a scheme according to which the product of two or more covariant normal-ordered quantities yields covariant normal-ordered results. We will discuss this point at length for the specific cases which occur in string theory.

Functionals

A functional $\Phi[q_i(\sigma)]$ is a scheme by which a (generally complex) number is associated with the quantities $q_i(\sigma)$; that is, the values of all of the q's are input (for all σ), and a number is the output. Such functionals are used in the Schrödinger representation of string quantum mechanics, and as string fields. Under reparametrizations, the functional will change due to the effect of the change in the arguments q_i; there may be additional terms in the transformation law for the field as well. If there are no such extra terms, the functional is referred to as a scalar functional. This is analogous to the transformation

of local fields under the Lorentz group: There is a term in the transformation law due to the change of the coordinates, and if the field is not a scalar then there will be additional (spin) terms. Generally any extra terms are allowed so long as the algebra is still satisfied.

For the special case of a scalar functional Φ, the change is given by

$$\delta_f(\epsilon)\Phi = -i\epsilon M_f \Phi \tag{7}$$

where the generators M_f have the form

$$M_f = -i\int_0^\pi d\sigma(fq_i' + w_i f' q_i + z_i f')\frac{\delta}{\delta q_i} \tag{8}$$

where each q_i transforms as (w_i, z_i). The M's satisfy the algebra

$$[M_f, M_g] = i M_{fg'-gf'} \tag{9}$$

An important consequence of this is that M must be built from an integrand which transforms covariantly with weight two. To see this, write M_g in the form

$$M_g = \int d\sigma g(\sigma) M(\sigma) \tag{10}$$

i.e. we have integrated by parts to remove all derivatives of the function g. Inserting this into (9) we find

$$\begin{aligned}\int d\sigma g(\sigma)\ [M_f, M(\sigma)] &= i\int d\sigma(fg' - f'g)M(\sigma)\\ &= -i\int d\sigma g(fM' + 2f'M)\end{aligned} \tag{11}$$

showing that the integrand M transforms with weight two. Alternatively one may see this from the explicit form

$$M(\sigma) = i\left[(w_i - 1)q_i'\frac{\delta}{\delta q} + (w_i q_i + z_i)\left(\frac{\delta}{\delta q}\right)'\right] \tag{12}$$

Super-reparametrizations

For superstring field theory we seek a kinematical supersymmetry transformation [41] $\not\delta_f$ which is the "square root" of the reparametrization δ_f in the sense that

$$[\not\delta_f(\xi_1), \not\delta_g(\xi_2)]F(\sigma) = -2\delta_{fg}(\xi_1\xi_2)F(\sigma) \tag{13}$$

for any field $F(\sigma)$. The ξ's are anticommuting parameters. The commutation relations of the $\not\delta$'s with the reparametrizations δ can be determined from the Jacobi identity

$$\Big[[\not\delta_f, \not\delta_g], \delta_h\Big] + \Big[[\delta_h, \not\delta_f], \not\delta_g\Big] + \Big[[\not\delta_g, \delta_h], \not\delta_f\Big] = 0. \tag{14}$$

We first note that the commutator of a reparametrization δ_h with a super-reparametrization $\not\delta_f$ must be bilinear in f, h and their derivatives; this is clear from (14) and (13). Furthermore, derivatives of order higher than one are excluded due to the presence of the first term in (14) (since this identity should hold for arbitrary functions). The commutator must therefore have the form

$$[\delta_h(\epsilon), \not\delta_f(\xi)] = \alpha\not\delta_{hf'+\beta h'f}(\epsilon\xi). \tag{15}$$

Using this relation and (13) in (14), we find that $\alpha = 1$ and $\beta = -\frac{1}{2}$, *i.e.*

$$[\delta_h(\epsilon), \not\delta_f(\xi)] = \not\delta_{hf'-h'f/2}(\epsilon\xi). \tag{16}$$

Henceforth, the parameters ϵ and ξ will not be indicated explicitly.

Representations of the Super-reparametrization Algebra

Given fields transforming in a specified manner under reparametrizations, we can deduce their possible transformation properties under super-reparametrizations. First consider the case of a field $a(\sigma)$, either commuting or anticommuting,

transforming covariantly under reparametrizations with weight w_a, for which we postulate the transformation law

$$\not\delta_f a = -fb \tag{17}$$

where b is a field of opposite type (commuting or anticommuting) from a. Equation (16) tells us that

$$-f\delta_g b = (\not\delta_f \delta_g - \not\delta_{gf'-g'f/2})a \tag{18}$$

and upon evaluating the right hand side of (18) we find

$$\delta_g b = -gb' - (w_a + \frac{1}{2})g'b \ , \tag{19}$$

i.e. b transforms covariantly with weight $w_b = w_a + \frac{1}{2}$. Assuming that a and b form a closed multiplet involving no other fields,* the most general form for the transformation of b under a super-reparametrization is

$$\not\delta_f b = \sum_n A_n \frac{d^n a}{d\sigma^n} \tag{20}$$

where the A_n's are functions of f and its derivatives. Using (13) with $g = f$, we find

$$\not\delta_f \not\delta_f b = -\delta_{ff} b \tag{21}$$

so that

$$-\sum_n A_n \frac{d^n}{d\sigma^n} fb = ffb' + 2w_b ff'b. \tag{22}$$

Since b and its derivatives are all independent, we can equate coefficients on either side to solve for the A_n's. We find that the only non-zero A_n's are $A_0 = -2w_a f'$ and $A_1 = -f$, i.e.,

$$\not\delta_f b = -(fa' + 2w_a f'a). \tag{23}$$

* In Appendix B we show that adding extra fields does not generate new irreducible representations.

We have discovered one type of multiplet on which the super-reparametrization algebra is represented. The representation can be written in matrix form:

$$\not\delta_f \begin{pmatrix} a \\ b \end{pmatrix} = -\begin{pmatrix} 0 & f \\ f\frac{d}{d\sigma} + 2w_a f' & 0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}, \tag{24}$$

whereas the transformation δ_f is written as

$$\delta_f \begin{pmatrix} a \\ b \end{pmatrix} = -\begin{pmatrix} f\frac{d}{d\sigma} + w_a f' & 0 \\ 0 & f\frac{d}{d\sigma} + (w_a + \frac{1}{2})f' \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \tag{25}$$

The representation is the same regardless of which field a or b is the commuting one. For this type of multiplet, we will refer to the component a transforming according to (17) as the light component, and to b which transforms according to (23) as the heavy component. An important difference between the two components is that if the integral of the heavy component is reparametrization invariant (*i.e.* if it has weight one), then it is also super-reparametrization invariant, as is evident from the transformation law (23). The integral over the light component is never super-reparametrization invariant.

An example of this type of representation is provided by the string coordinates x^μ. These transform according to (17) into generalized Dirac matrices [23] Γ^μ:

$$\not\delta_f x^\mu = -f\Gamma^\mu \tag{26}$$

$$\not\delta_f \Gamma^\mu = -f x^{\mu\prime} \tag{27}$$

Because x^μ has weight zero, the multiplet (Γ^μ, x'^μ) also transforms as (24), with Γ^μ as the light component. This multiplet is of more direct use in string field theory because it is translationally invariant.

Direct Product Representations

Given two doublets (a, b) and (c, d), we again wish to know all of the different covariant super-reparametrization representations which can be built out of products of these fields and their derivatives. One can form eight quantities which transform covariantly:

$$\begin{aligned}
&\text{weight } w: && A_1 = ac \\
&\text{weight } w+\frac{1}{2}: && A_2 = ad \ \text{ and } A_3 = bc \\
&\text{weight } w+1: && A_4 = bd \ \text{ and } A_5 = w_c a' c - w_a a c' \\
&\text{weight } w+\frac{3}{2}: && A_6 = (w_c + \frac{1}{2}) a' d - w_a a d' \ \text{ and } A_7 = w_c b' c - (w_a + \frac{1}{2}) b c' \\
&\text{weight } w+2: && A_8 = (w_c + \frac{1}{2}) b' d - (w_a + \frac{1}{2}) b d'
\end{aligned} \tag{28}$$

In these equations, $w \equiv w_a + w_c$. Among these quantities, three combinations may be identified as doublets:

$$(A_1, \pm A_2 + A_3), \qquad \text{with weight } (w, w+\frac{1}{2}) \tag{29a}$$

$$(w_a A_2 \mp w_c A_3, \pm A_5 + w A_4), \qquad \text{with weight } (w+\frac{1}{2}, w+1) \tag{29b}$$

$$(\pm 2A_5 + A_4, \pm 2A_7 + 2A_6), \qquad \text{with weight } (w+1, w+\frac{3}{2}) \tag{29c}$$

The upper (lower) sign of the $\pm$'s in these equations is to be read in the case where a is the commuting (anticommuting) member of its multiplet. The heavy component of both (29 b) and (29 c) reduce to total derivatives in the cases in which their weight is one, so they yield only trivial invariants. The remaining two quantities in (28) are members of a multiplet containing non-covariant quantities.

We have thus demonstrated the decomposition

$$\mathbf{2_w} \otimes \mathbf{2_v} = \mathbf{2_{v+w}} \oplus \mathbf{2_{v+w+\frac{1}{2}}} \oplus \mathbf{2_{v+w+1}} \oplus (\text{non}-\text{covariant}) \tag{30}$$

The fact that the only covariant representation of the super-reparametrization algebra found in the direct product of two doublets is again a doublet suggests that no other covariant irreducible representations exist. This is true, and a detailed proof is given in Appendix B.

FREE FIELD THEORY

In this section we apply the reparametrization group theory developed in the previous section to the determination of the Schrödinger equation for free strings, and from that the equation of motion and action of free string field theory. Although the results are very well known, the details of the construction may not be.

Second Quantization

The free quantum mechanics of strings will be formulated in terms of the wave functional $\Psi[x]$. This functional obeys a Schrödinger equation,

$$Q\Psi[x] = 0 \tag{31}$$

for some operator Q. Also of interest in quantum mechanics is the inner product of two wave functionals, given by

$$< \Psi_1|\Psi_2 >= \int Dx \Psi_1^*[x]\Psi_2[x] \tag{32}$$

The Schrödinger equation (31) will become the classical equation of motion for the free string field. The string field generally resembles the wave functional except that there may be a reality condition on the string field [42]. However, this reality condition is irrelevant at the free level due to the linearity of the equation of motion. We will return to this point later. The action functional for the free field theory is then given by

$$S\{\Phi\} =< \Phi|Q|\Phi >= \int Dx \Phi^*[x]Q\Phi[x] \tag{33}$$

On physical grounds we would expect the quantum theory to be invariant under the global reparametrization symmetry. We will take the string wave functional to be a scalar, meaning that its change under reparametrizations is solely that due to the change in the coordinates. Under $\sigma \rightarrow \sigma + \epsilon f(\sigma)$, we find

$$\delta\Psi[x] = M_f^x \Psi[x] \tag{34}$$

where

$$M_f^x \Psi \equiv -i \int_0^\pi d\sigma f(\sigma) x'(\sigma) \cdot \frac{\delta}{\delta x(\sigma)} \tag{35}$$

Reparametrization invariance of the quantum theory, and hence the free field theory, requires two conditions: For any solution Ψ of the Schrödinger equation, the reparametrized wave functional $\Psi + \delta\Psi$ must also be a solution; and the reparametrization operation must be hermitian with respect to the inner product. The second condition is already met by the form (35); the first will be met if and only if M_f commutes with Q for all functions f.

We might expect that the next step is to try to build Q using the coordinates x and their functional derivatives. This was the approach of Marshall and Ramond [23]. However, this approach is problematic. With great hindsight, we may understand the difficulty as follows: Physically, we would expect a string's wave functional to associate a number with each distinct string configuration. However, the functionals we use are functionals of parametrized strings; strings which differ by reparametrization are treated as distinct. This may not cause problems in the classical theory, but in the quantum theory we will wish to perform path integrals over the space of all possible string functionals; counting reparametrized strings as distinct will cause us to miscount in the integral. We would need to find a measure which is reparametrization invariant, which seems to be impossible to do using the string coordinates alone.

Thus the functionals of parametrized strings should be used only for a fixed parametrization.

More practically, the only nontrivial invariant operators which may be formed using only the string coordinates x are made using $\sqrt{x'(\sigma)^2}$, which is not a well-behaved operator in the string Hilbert space. We must therefore seek an alternative formulation of the theory if we wish to have reparametrization invariant equations of motion.

The Einbein

Reasoning by analogy with gravity, many authors [43,44,45,46,47] have realized that the dynamics of point particles can be reformulated by introducing an einbein. Indeed they have shown that the free point particle action ($\dot{x} \equiv \frac{dx}{d\tau}$)

$$m \int_{\tau_i}^{\tau_f} d\tau \sqrt{\dot{x}^2} \tag{36}$$

can be replaced by

$$\int d\tau \left[\frac{1}{e(\tau)} \dot{x}^2 + m^2 e(\tau) \right], \tag{37}$$

where $e(\tau)$ is an einbein transforming into a total derivative under reparametrizations of the proper time parameter τ:

$$\delta e(\tau) = -\epsilon \frac{d}{d\tau} \left[f(\tau) e(\tau) \right]$$

This action led to the same classical dynamics as the previous one, at the cost of introducing a new quantity. We are interested in building reparametrization invariant quantities; their construction shows how to generalize our expressions while avoiding square roots. Thus consider [40] introducing a variable $e(\sigma)$ transforming covariantly with some weight w

$$\delta e(\sigma) = -\epsilon [f(\sigma) e'(\sigma) + w f'(\sigma) e(\sigma)] \tag{38}$$

which is the representation $(w, 0)$ of the last section. However, the functional derivative $\frac{\delta}{\delta e(\sigma)}$ transforms with weight $1 - w$, so its integral is in general no longer invariant. The more complicated operator $e\frac{\delta}{\delta e} = \frac{\delta}{\delta \ln e}$ does transform with weight one, so its integral is invariant. So we make the change of variables [48] to

$$\phi(\sigma) = \ln e(\sigma) \tag{39}$$

This transforms inhomogeneously, as $(0, w)$:

$$\delta\phi(\sigma) = -\epsilon f(\sigma)\frac{d\phi}{d\sigma} - \epsilon w \frac{df}{d\sigma} \tag{40}$$

so its reparametrization generators are given by

$$M_f^\phi = -i\int d\sigma \left[f(\sigma)\frac{d\phi}{d\sigma} + w\frac{df(\sigma)}{d\sigma}\right]\frac{\delta}{\delta\phi(\sigma)} \tag{41}$$

The quantity ϕ will be familiar as the bosonized ghost of string theory.

Mode Expansions

We can expand the various fields in Fourier modes. For the string coordinates x^μ, the open string boundary conditions demand that $x'^\mu = 0$ at the endpoints $\sigma = 0, \pi$. Thus x^μ has an expansion purely in terms of cosines:

$$x^\mu(\sigma) = x_0^\mu + \sqrt{2}\sum_{n=1}^{\infty} x_n^\mu \cos n\sigma, \tag{42}$$

whereas the functional derivative $\frac{\delta}{\delta x}$ has the expansion

$$\frac{\delta}{\delta x^\mu(\sigma)} = \frac{\partial}{\partial x_0^\mu} + \sqrt{2}\sum_{n=1}^{\infty}\frac{\partial}{\partial x_n^\mu}\cos n\sigma. \tag{43}$$

In terms of creation and annihilation operators (a background-dependent decomposition)

$$\alpha_n^\mu = -i\sqrt{2}\left(\frac{\partial}{\partial x_{n\mu}} + n x_n^\mu\right) \tag{44}$$

$$\alpha^\mu_{-n} = -i\sqrt{2}\left(\frac{\partial}{\partial x_{n\mu}} - n x^\mu_n\right) \tag{45}$$

$$x^\mu_n = \frac{i}{n\sqrt{2}}[\alpha^\mu_n - \alpha^\mu_{-n}]$$

$$\frac{\partial}{\partial x^\mu_n} = \frac{i}{\sqrt{2}}[\alpha_{n\mu} + \alpha_{-n\mu}] \tag{46}$$

the expansion of x^μ and of the functional derivatives become, respectively,

$$x^\mu(\sigma) = x^\mu_0 + i\sum_{n=1}^{\infty}\frac{1}{n}[\alpha^\mu_n - \alpha^\mu_{-n}]\cos n\sigma \tag{47}$$

and

$$\frac{\delta}{\delta x_\mu} = i\alpha^\mu_0 + i\sum_{n=1}^{\infty}[\alpha^\mu_n + \alpha^\mu_{-n}]\cos n\sigma. \tag{48}$$

The α's satisfy

$$[\alpha^\mu_m, \alpha^\nu_n] = m g^{\mu\nu}\delta_{m+n,0} \tag{49}$$

and

$$[x^\mu_0, \alpha^\nu_0] = i g^{\mu\nu}.$$

We can similarly expand the field ϕ,

$$\begin{aligned} \phi(\sigma) &= \phi_0 + i\sum_{n=1}^{\infty}\frac{1}{n}(\beta_n - \beta_{-n})\cos n\sigma \\ \frac{\delta}{\delta\phi} &= i\eta\sum_{n=-\infty}^{\infty}\beta_n\cos n\sigma \end{aligned} \tag{50}$$

where

$$[\beta_m, \beta_n] = \eta m\delta_{m,-n} \tag{51}$$

$$[\phi_0, \beta_m] = i\eta\delta_{m,0} \tag{52}$$

and $\eta = \pm 1$. The ground state Φ_0 of the representation is defined to satisfy $\alpha_m\Phi_0 = \beta_m\Phi_0 = 0$ for all $m > 0$. The mode operators β_m create states of positive or negative norm depending on the choice for η. Expanding the generators M in modes shows the need to redefine M when the full quantum

nature of the string Hilbert space is taken into account. Since $x'(\sigma)$ does not commute with $\frac{\delta}{\delta x(\sigma)}$, their product has a possible ordering ambiguity. In such a case, we need a convention to fix the ordering. The normal ordered product is defined to be the ordering in which all positive modes α_n are on the right, and all negative modes α_{-n} on the left. The new normal ordered M differs by a finite correction:

$$M_f =: M_f^{Classical} : \tag{53}$$

These generators are related to a subgroup of the Virasoro algebra:

$$M_{\sqrt{2}\sin n\sigma} = -\frac{i}{\sqrt{2}}(L_n - L_{-n}) \tag{54}$$

and

$$M^{Classical}_{\sqrt{2}\sin n\sigma} = -\frac{i}{\sqrt{2}}(L_n - L_{-n} + C_n) \tag{55}$$

where L_n are the well-known Virasoro operators,

$$L_n = L_n^x + L_n^\phi \tag{56}$$

$$L_n^x \equiv \frac{1}{2}\sum_l \alpha_l \cdot \alpha_{n-l} \tag{57}$$

$$L_n^\phi \equiv \frac{1}{2}\sum_l \beta_l \beta_{n-l} + i\frac{wn}{2}\beta_n \tag{58}$$

and

$$C_n = \begin{cases} \frac{n(D+1)}{8} & \text{for } n \text{ even,} \\ 0 & \text{for } n \text{ odd.} \end{cases} \tag{59}$$

Note that the redefinition of M_f to remove the c-number anomaly term C_n affects neither the algebra (9) nor the free invariance condition $[Q, M_f] = 0$.

Left and Right Combinations

The generators (35) can be rewritten in terms of the commuting combinations $\left(x' \pm i\pi\frac{\delta}{\delta x}\right)^2$:

$$M_f = M_f^L + M_f^R$$

where

$$M_f^L = -\frac{1}{4}\int \frac{d\sigma}{\pi} f\left(x' + i\pi\frac{\delta}{\delta x}\right)^2 \tag{60}$$

and

$$M_f^R = +\frac{1}{4}\int \frac{d\sigma}{\pi} f\left(x' - i\pi\frac{\delta}{\delta x}\right)^2 . \tag{61}$$

This split-up is familiar from first-quantized string theory as the splitting of the string coordinate into left and right movers; we identify $x'_L \equiv x' + i\pi\frac{\delta}{\delta x}$ as the derivative of the left-moving part of the string coordinate, and $x'_R \equiv x' - i\pi\frac{\delta}{\delta x}$ as that of the right-moving part. This identification is confirmed by the mode expansions in terms of exponentials

$$x_L^\mu(\sigma) = x_0^\mu - \sigma\alpha_0^\mu + i\sum_{n\neq 0}\frac{\alpha_n^\mu}{n}e^{in\sigma} \tag{62}$$

and

$$x_R^\mu(\sigma) = x_L^\mu(-\sigma) \tag{63}$$

which agree with the conventions of the first-quantized theory. [5,3,4]. Note that this split-up would not be possible in the presence of a curved space-time metric $g_{\mu\nu}$.

The M_f generators for the field ϕ split in a similar manner, with

$$\phi'_L = \phi' + i\eta\frac{\delta}{\delta\phi} \tag{64}$$

and

$$\phi'_R = \phi' - i\eta\frac{\delta}{\delta\phi}. \tag{65}$$

so

$$M_f = -\frac{\eta}{2}\int_0^\pi \frac{d\sigma}{\pi}\left(f\frac{{\phi'_L}^2 - {\phi'_R}^2}{2} + wf'(\phi'_L - \phi'_R)\right) \tag{66}$$

Because of this split-up, we expect that it should be possible to build covariant quantities purely out of left or right moving fields alone. For example, the

normal-ordered exponential $:e^{a\phi_L}:$ can be seen [40] to transform covariantly with weight $a(w-\eta a/2)$.

The left and right combinations commute with each other; the relation (9) then requires that

$$[M_f^L, M_g^L] + [M_f^R, M_g^R] = i(M_{fg'-f'g}^L + M_{fg'-f'g}^R). \tag{67}$$

Since M^L contains only left pieces, and M^R only right ones, and left movers commute with right movers, (67) can be separated into left and right pieces, which must each be zero up to a c-number:

$$\begin{aligned}[M_f^L, M_g^L] &= iM_{fg'-f'g}^L + c_{f,g}\\ [M_f^R, M_g^R] &= iM_{fg'-f'g}^R - c_{f,g}\end{aligned} \tag{68}$$

To evaluate the c-number we must expand in Fourier modes and carefully take into account ordering effects; we shall do so in the next subsection. However, the form of the c-number may be deduced from algebraic properties alone. Applying the Jacobi identity for the commutator of three M^L generators yields a relation that the c-number term must satisfy:

$$c_{f,gh'-hg'} + c_{g,hf'-fh'} + c_{h,fg'-gf'} = 0. \tag{69}$$

Additionally, by the antisymmetry of the commutator,

$$c_{f,g} + c_{g,f} = 0. \tag{70}$$

Since the generators M_f are linear in f and its derivatives, the most general form for the c-number $c_{f,g}$ is

$$c_{f,g} = \int \frac{d\sigma}{\pi} \sum_{n=0}^{\infty} c_n g(\sigma) \frac{d^n}{d\sigma^n} f(\sigma) \tag{71}$$

where we have integrated by parts to remove all derivatives of f. The constraint (70) is satisfied if and only if $c_n = 0$ for all even n. The constraint (69) (after integration by parts) requires

$$\int \frac{d\sigma}{\pi} \sum_n c_n \left[(-1)^n \frac{d^n}{d\sigma^n}(gh' - hg') - 2h' \frac{d^n g}{d\sigma^n} \right.$$

$$\left. -h \frac{d^{n+1} g}{d\sigma^{n+1}} + 2g' \frac{d^n h}{d\sigma^n} + g \frac{d^{n+1} h}{d\sigma^{n+1}} \right] f = 0 \tag{72}$$

which is satisfied if and only if $n = 0, 1$, or 3. Thus the most general form which satisfies both constraints is [49]

$$c_{f,g} = \int \frac{d\sigma}{\pi}(c_1 f' g + c_3 f''' g). \tag{73}$$

An advantage of the left-right split-up is that the operator

$$\Box_f \equiv M_f^L - M_f^R \tag{74}$$

is automatically a covariant dynamical operator of weight two, provided that the anomaly $c_{f,g}$ is zero. With the fields x^μ and ϕ this operator has the form

$$\Box_f = \frac{1}{2} \int \frac{d\sigma}{\pi} f : \left(\pi^2 \frac{\delta^2}{\delta x^2} - x'^2 - \pi^2 \frac{\delta^2}{\delta \phi^2} + \phi'^2 - 2w\phi'' \right) : \tag{75}$$

This operator contains second-order time derivatives, hence it is dynamical, unlike the purely kinematical reparametrization operator M_f; because of this, it is referred to as a generalized Klein-Gordon operator. The split-up of M into M_L and M_R is of great physical significance as $M_L + M_R$ contains no time derivatives and also does not depend on the metric of the background space.

Anomaly Cancellation

We now evaluate the c-number anomaly term in (68). This may be found directly by expanding the generators in modes, or by evaluating the product

$M^L(\sigma_1)M^L(\sigma_2)$, where $\sigma_1 \approx \sigma_2$, as a series of normal-ordered operators; the anomaly c_3 will be given by the c-number term in the series. We will show this approach to illustrate the technique. Recall

$$M^L(\sigma) = {x'_L}^2 + \eta {\phi'_L}^2 + 2w\eta\phi'' \tag{76}$$

Using Wick's theorem, the c-number term is given by

$$\begin{aligned} c = 2 &< {x'}_{L\mu}(\sigma_1), {x'}_{L\nu}(\sigma_2) >< {x'_L}^{\mu}(\sigma_1), {x'_L}^{\nu}(\sigma_2) > \\ &+ 2 < \phi'_L(\sigma_1), \phi'_L(\sigma_2) >^2 + 4w^2 < \phi''_L(\sigma_1), \phi''_L(\sigma_2) > \end{aligned} \tag{77}$$

where the correlation functions are defined as, e.g.,

$$\begin{aligned} < \phi'_L(\sigma_1), \phi'_L(\sigma_2) > &\equiv < 0|\phi'_L(\sigma_1)\phi'_L(\sigma_2)|0 > \\ &= \phi'_L(\sigma_1)\phi'_L(\sigma_2) - : \phi'_L(\sigma_1), \phi'_L(\sigma_2) : \end{aligned} \tag{78}$$

These may be evaluated directly:

$$\begin{aligned} < \phi'_L(\sigma_1), \phi'_L(\sigma_2) > &= < 0|\sum_n \sum_m (\beta_n e^{in\sigma_1})(\beta_m e^{im\sigma_2})|0 > \\ &= < \beta_0^2 > + \eta \sum_{n=1}^{\infty} n e^{in(\sigma_1 - \sigma_2) - n\epsilon} \end{aligned} \tag{79}$$

where $\epsilon > 0$ has been included for convergence. We then sum the series and evaluate it for $\sigma_1 \approx \sigma_2$. The result is

$$< \phi'_L(\sigma_1), \phi'_L(\sigma_2) > = \frac{\eta}{(\sigma_1 - \sigma_2 + i\epsilon)^2} \tag{80}$$

Then the total anomaly is

$$c = \left(2(D+1) + 4w^2 \cdot 6\eta\right)\left(\frac{1}{\sigma_1 - \sigma_2 + i\epsilon}\right)^4 \tag{81}$$

It is clear that we must have $\eta = -1$, otherwise the anomaly can only cancel if $D + 1 < 0$, not physically reasonable. Then the anomaly is found to cancel for

$$D = 12w^2 - 1 \tag{82}$$

Construction of the Invariant Operator

Finally, we are now ready to construct an invariant free field equation. The quantity $M^L(\sigma)$ (the integrand of M_f) has the right dynamical structure (e.g. it contains $\frac{\delta^2}{\delta x^2}$), but it is of weight two; therefore we must multiply by a measure of weight -1, which is provided by the exponential $: e^{a\phi_L} :$.

When $\eta = -1$, the invariant operator is given by

$$Q \equiv -\int_0^\pi \frac{d\sigma}{\pi} : e^{a\phi_L(\sigma)} M^L(\sigma) : + (L \to R) \tag{83}$$

where $a = \pm 1, \pm 2$ so that the exponential has weight -1. The operator Q is invariant under reparametrizations. It acts in a Fock space with a spectrum bounded from below provided that it can be written in a normal ordered form. Recall that in terms of modes, we have

$$M_n \equiv M_{e^{in\sigma}} = L_n - L_{-n},$$

with

$$L_n = \frac{1}{2} \colon \sum_{k=0}^{\infty} (\alpha_{n-k} \cdot \alpha_k + \eta \beta_{n-k} \beta_k) : + \eta i w n \beta_n. \tag{84}$$

We need to be check that overall normal ordering does not affect the invariance of this operator. We find that

$$[L_n, Q] = \frac{n(n+1)}{2} \int_0^\pi \frac{d\sigma}{\pi} e^{-in\sigma} (i\frac{d}{d\sigma} - \frac{a^2}{2} - \frac{aw}{3} - n\frac{2aw}{3}) : e^{a\phi_L} : + (L \to R). \tag{85}$$

The integrand is a total derivative iff. $2aw = -3$ and $a^2 = 1$, the contribution at the lower limit cancelling between left and right pieces. The only solutions are $a = \pm 1$ corresponding to $w = \mp \frac{3}{2}$ respectively. These solutions are of course equivalent since redefining $\phi \to -\phi$ would change from $w = +\frac{3}{2}$ to $w = -\frac{3}{2}$; we will make the latter choice so that $a = +1$. This gives us a unique invariant operator whose construction, by (82), is only possible in 26

dimensions. This of course is the usual BRST charge, and one can check that Q is in fact nilpotent. Also, if $w = -\frac{3}{2}$, while $: e^{\phi_L(\sigma)} :$ has weight -1, the other combination $: e^{-\phi_L(\sigma)} :$ has weight two; it anticommutes with $: e^{\phi_L(\sigma)} :$ to give a delta function, and thus can be taken to be the conjugate antighost field b_{++} corresponding to c^+ which is $: e^{\phi_L(\sigma)} :$ [5].

Due to the nilpotency of Q, if Φ is a solution of the free field equation $Q\Phi = 0$, then $\Phi + Q\Lambda$ will also be a solution, where Λ is an arbitrary functional. This is a gauge invariance of the string field theory which insures that the negative norm states of the first-quantized theory decouple from the physical states [34].

CATALOG OF INVARIANTS

In this section we digress to list and discuss the various reparametrization-invariant operators present in the bosonic theory. These are found by the simple group- theoretical procedure described previously, i.e. to combine covariant quantities using the direct product rules for representations, and to check that normal ordering does not affect the invariance. We find many familiar results, along with some surprises. The list is:

(1) The momentum vector

$$p_\mu \equiv -i \int_0^\pi d\sigma \frac{\delta}{\delta x^\mu(\sigma)}. \tag{86}$$

(2) The ghost number

$$N_G \equiv -i \int_0^\pi d\sigma \frac{\delta}{\delta \phi(\sigma)}. \tag{87}$$

(3) The Lorentz generators

$$M_{\mu\nu} \equiv i \int_0^\pi d\sigma \left(g_{\mu\rho} x^\rho \frac{\delta}{\delta x^\nu} - g_{\nu\rho} x^\rho \frac{\delta}{\delta x^\mu} \right) \tag{88}$$

(4) The symmetric space-time tensor

$$Q^{\mu\nu} \equiv \int_0^\pi \frac{d\sigma}{\pi} : e^{\phi_L} \left[x_L^{\prime\mu} x_L^{\prime\nu} - \frac{g^{\mu\nu}}{26} (\phi_L^{\prime 2} + 3\phi_L^{\prime\prime}) \right] : +(L \to R). \tag{89}$$

This invariant tensor depends on the space-time geometry. The BRST operator is obtained by taking its its trace:

$$Q = g_{\mu\nu} Q^{\mu\nu}. \tag{90}$$

(5) An additional symmetric tensor

$$B^{\mu\nu} = \int_0^\pi \frac{d\sigma}{\pi} : e^{2\phi_L} (x_L^{\prime\mu} x_L^{\prime\nu} - \frac{g^{\mu\nu}}{26} x'_L \cdot x'_L) : +(L \to R). \tag{91}$$

We note that although the dilatation operator $D = \int d\sigma : x \cdot \frac{\delta}{\delta x} :$ has the right weight to be a classical invariant, it transforms anomalously due to ordering effects. Thus the largest space-time symmetry seems to be that of the Poincaré group. We remark that there does not exist an invariant 26-vector which serves as the string position in space-time. This is not too surprising since the theory is not (space-time) conformally invariant. On the other hand, by specializing the Poincaré generators to the relevant space-like surfaces, we can define a physical position for the string in 25 (at equal time) or 24 (light cone) space dimensions.

Operator Algebras

All of the operators we have described are normal ordered and invariant under the complex reparametrization algebra. We proceed to study the algebra obeyed by these special quantities [50]. P_μ and $M_{\mu\nu}$ form the algebra of the Poincaré group, and commute with ghost number, while $Q^{\mu\nu}$ has ghost number 1 and $B^{\mu\nu}$ ghost number 2:

$$[N, Q^{\mu\nu}] = Q^{\mu\nu} \ ; \ [N, B^{\mu\nu}] = 2B^{\mu\nu}. \tag{92}$$

Furthermore, the translationally invariant $Q^{\mu\nu}$ and $B^{\mu\nu}$ transform as second rank tensors under the Lorentz group.

On the other hand, the $Q^{\mu\nu}$ obey a more complicated anticommuting algebra:

$$\begin{aligned}\{Q^{\mu\nu}, Q^{\rho\sigma}\} = &- i(g^{\nu\rho}B^{\mu\sigma} + g^{\mu\rho}B^{\nu\sigma} + g^{\mu\sigma}B^{\nu\rho} + g^{\nu\sigma}B^{\mu\rho}) \\ &+ \frac{2}{13}[(g^{\mu\nu}B^{\rho\sigma} + g^{\rho\sigma}B^{\mu\nu}) - (g^{\mu\sigma}g^{\nu\rho} + g^{\mu\rho}g^{\nu\sigma} - \frac{1}{13}g^{\mu\nu}g^{\rho\sigma})C] \, ,\end{aligned} \tag{93}$$

where

$$C = \int_0^\pi \frac{d\sigma}{\pi} e^{2\phi_L} x'_L \cdot x'_L \ + (L \to R); \tag{94}$$

it is an invariant tensor, but it is *not* normal ordered. Hence the algebra of the $Q^{\mu\nu}$ does not close on normal-ordered invariant operators, which is hardly surprising. The algebra is completed by noting that $B^{\mu\nu}$ commutes with itself and with $Q^{\mu\nu}$

$$[B^{\mu\nu}, Q^{\rho\sigma}] = [B^{\mu\nu}, B^{\rho\sigma}] = 0. \tag{95}$$

In order to determine the largest non-anomalous algebra among our operators, let us introduce a set of 26×26 matrices ${\alpha^I}_{\mu\nu}$ such that the projections

$$Q^I \equiv {\alpha^I}_{\mu\nu} Q^{\mu\nu} \equiv \mathrm{Tr}(\alpha^I \mathbf{Q}) \tag{96}$$

satisfy a non-anomalous algebra. It is easy to see that

$$\begin{aligned} \{Q^I, Q^J\} =& 2i\{\alpha^I, \alpha^J\}_{\mu\nu} B^{\mu\nu} + \frac{i}{13}\{B^J \mathrm{Tr}(\alpha^I) + B^I \mathrm{Tr}(\alpha^J)\} \\ & - \frac{i}{13}\{\mathrm{Tr}(\alpha^I \alpha^J) - \frac{1}{26}\mathrm{Tr}(\alpha^I)\mathrm{Tr}(\alpha^J)\}C, \end{aligned} \tag{97}$$

where $B^I = {\alpha^I}_{\mu\nu} B^{\mu\nu}$. Thus the algebra will not be anomalous as long as we require

$$\mathrm{Tr}(\alpha^I \alpha^J) = \frac{1}{26}\mathrm{Tr}(\alpha^I)\mathrm{Tr}(\alpha^J) \ , \tag{98}$$

where the trace operation is meant to act between covariant and contravariant indices, *i.e.*

$$\mathrm{Tr}\ \alpha^I = (\alpha^I)^\mu{}_\mu \ . \tag{99}$$

This condition is obeyed by the metric itself

$$\mathrm{Tr}(\mathrm{g}\ \mathrm{g}) = \frac{1}{26}\mathrm{Trg}\ \mathrm{Trg} \ . \tag{100}$$

In fact this gives the only Lorentz-invariant projection,

$$Q = g_{\mu\nu} Q^{\mu\nu}; \tag{101}$$

this is the nilpotent BRST charge:

$$\{Q, Q\} = 0 \ . \tag{102}$$

Similarly, we see that $B^{\mu\nu}$ is the BRST transform of $Q^{\mu\nu}$

$$\{Q, Q^{\mu\nu}\} = 2iB^{\mu\nu} \ , \tag{103}$$

so that it commutes with Q.

We can redefine the matrices α^I by subtracting out their traces:

$$\alpha^I_{\mu\nu} \rightarrow \alpha^I_{\mu\nu} - g_{\mu\nu} \operatorname{Tr} \alpha^I \tag{104}$$

Thus it is enough to concentrate on the redefined set of 26×26 matrices which now satisfy the following conditions

$$\begin{gathered} (\alpha^I)_{\mu\nu} = (\alpha^I)_{\nu\mu} \ ; \ (\alpha^I)^{\mu}{}_{\mu} = 0 \ ; \\ (\alpha^I)_{\mu}^{\rho} (\alpha^J)_{\rho}^{\mu} = 0 \ . \end{gathered} \tag{105}$$

The traceless projections of Q obey the algebra

$$\{Q^I, Q^J\} = 2i\{\alpha^I, \alpha^J\}_{\mu\nu} B^{\mu\nu} \ . \tag{106}$$

These matrices α^I do not exist in Euclidean space. In D-dimensional Minkowski space, K, the number of linearly independent matrices obeying these properties, can be written as the number of symmetric traceless matrices less the number of constraints, *i.e.*

$$K = \frac{D(D+1)}{2} - 1 - \frac{K(K+1)}{2} \ , \tag{107}$$

which is satisfied when $K = D - 1$. Thus, including the Lorentz invariant BRST charge, there are at most 26 linearly independent Q^I which satisfy a non-anomalous algebra.

None of these, except the trace, commutes with the full Lorentz group. These projections can be transformed by Lorentz transformations into other operators which are not in the set $\{Q^I\}$ since the Lorentz algebra has been broken by the very choice of the α^I matrices. Hence, it is only in the context of a broken Lorentz algebra that the operators Q^I need to be considered. Manifest Lorentz invariance yields an algebra of one extra operator, the BRST operator of the usual open string field theory. This suggests that we try to arrange the inequivalent sets of α matrices in terms of the unbroken sub-Lorentz algebras. If there were a set in which the Lorentz algebra were broken to a 3+1 dimensional algebra, it would possibly be of physical relevance, since the world of experience indicates only Lorentz invariance in four dimensions, not in 26 dimensions.

In general, the set of α matrices is closed under linear combinations; its structure is that of a linear vector space [51]. An arbitrary element of this space can be written in the form

$$\alpha^{\mu}{}_{\nu} = \begin{pmatrix} -\mathrm{Tr}\mathbf{A} & \mathbf{a}^T \\ -\mathbf{a} & \mathbf{A} \end{pmatrix} \tag{108}$$

where $\mathbf{a}$ is a 25−vector and $\mathbf{A}$ is a symmetric 25×25 matrix. The constraints (105) imply

$$\mathrm{Tr}(\mathrm{A}^I \mathrm{A}^J) + \mathrm{Tr}\mathrm{A}^I \mathrm{Tr}\mathrm{A}^J = 2\mathrm{a}^I \cdot \mathrm{a}^J \ . \tag{109}$$

Thus the quantity

$$(\alpha^I, \alpha^J) \equiv \frac{1}{2}\Big(\mathrm{Tr}(\mathrm{A}^I \mathrm{A}^J) + \mathrm{Tr}\mathrm{A}^I \mathrm{Tr}\mathrm{A}^J\Big) = \mathrm{a} \cdot \mathrm{a} \tag{110}$$

is an inner product on the space of α's. We can choose a basis in this space which is orthonormal with respect to this inner product. Since only $D-1$

orthonormal vectors **a** may be found, we see again that the maximum number of α matrices is $D-1$.

If a particular α^I is nilpotent , so is the corresponding operator Q^I, and we can show that there is at most one nilpotent matrix for each set of the matrices. Nilpotency of a given α matrix implies the equations

$$\begin{aligned} \mathrm{A}^2 &= \mathbf{a}\ \mathbf{a}^T \\ \mathrm{A}\mathbf{a} &= (\mathrm{Tr}\mathrm{A})\mathbf{a}\ . \end{aligned} \tag{111}$$

These equations can be solved by introducing the null 26-dimensional vector a_μ with components $(\mathrm{Tr}\mathrm{A}, \mathbf{a})$, in terms of which we have

$$\alpha_{\mu\nu} = a_\mu a_\nu\ ;\quad a_\mu a^\mu = 0. \tag{112}$$

Let $\alpha'_{\mu\nu} = b_\mu b_\nu$ be another nilpotent matrix ($b_\mu b^\mu = 0$). We see that $\mathrm{Tr}(\alpha\alpha') = 0$ implies that $a_\mu b^\mu = 0$, so that a_μ and b_μ must be proportional to one another: there is only one nilpotent per set, besides the Lorentz invariant BRST nilpotent. (One can also show that there is no nilpotent linear combination of Q and Q^I; otherwise it would imply

$$\alpha^I(\alpha^I + 1) = 0, \tag{113}$$

meaning that α^I has only 0 or -1 as eigenvalues. Since it is to be traceless, it can only have zero eigenvalues, and thus is similar to a triangular matrix, which contradicts its pseudosymmetry.)

It is not very hard, albeit tedious, to produce sets of singular α matrices by inductive procedures. For instance, if we know a set of such matrices in $D-1$ dimensions, we can insert them in a set of $D \times D$ matrices with the Dth column and row being zeros, and the remaining α matrix will have zeros in its first $D-1$ rows and columns, and a null vector in its Dth row and column.

The eigenvalues of the α matrices are in general complex numbers forming conjugate pairs. In D dimensions, they are determined by $2D-5$ parameters and one overall normalization. There can be at most 7 commuting elements in the non-singular set in 26 dimensions, although this number will increase whenever singular elements are present.

An Example

We now give a specific example. It is very simple and serves to illustrate our procedure, which up to now has been of a general nature. Let a_μ and b_μ be two null vectors with the properties

$$a^\mu a_\mu = 0 \; ; \; b^\mu b_\mu = 0 \; ; \; a^\mu b_\mu = 1 \; , \tag{114}$$

and introduce a set of 24 transverse orthonormal vectors $c^{(i)}_\mu$ which satisfy

$$c^{(i)}_\mu a^\mu = c^{(i)}_\mu b^\mu = 0 \; ; \; c^{(i)}_\mu c^{(j)\mu} = -\delta^{ij} \; . \tag{115}$$

Then it is easy to see that a set of α matrices can be formed out of these according to

$$\alpha^{(o)}_{\mu\nu} = a_\mu a_\nu \; , \tag{116a}$$

$$\alpha^{(i)}_{\mu\nu} = a_\mu c^{(i)}_\nu + a_\nu c^{(i)}_\mu \; , \quad i = 1, \ldots, 24 \; . \tag{116b}$$

This is a rather degenerate example, since all these matrices are singular. The ensuing projections Q^o and Q^i satisfy the superalgebra

$$\begin{aligned} \{Q^o, Q^o\} &= 0 \; , \\ \{Q^o, Q^i\} &= 0 \; , \\ \{Q^i, Q^j\} &= -2i\delta^{ij} B^o \; . \end{aligned} \tag{117}$$

The algebra can be enlarged to include the BRST charge, Q, with the results

$$\begin{aligned} \{Q, Q^o\} &= 2iB^o \ , \\ \{Q, Q^i\} &= 2iB^i \ . \end{aligned} \tag{118}$$

The relevant projections of the Lorentz generators which map the set into itself are obtained through the antisymmetric light-cone matrices

$$\begin{aligned} m^{+-}_{\mu\nu} &= a_\mu b_\nu - a_\nu b_\mu \ , \\ m^{+i}_{\mu\nu} &= a_\mu c^{(i)}_\nu - a_\nu c^{(i)}_\mu \ , \\ m^{-i}_{\mu\nu} &= b_\mu c^{(i)}_\nu - b_\nu c^{(i)}_\mu \ , \\ m^{ij}_{\mu\nu} &= c^{(i)}_\mu c^{(j)}_\nu - c^{(j)}_\mu c^{(i)}_\nu \ . \end{aligned} \tag{119}$$

It is easy to see that the only subalgebra which keeps the set within itself is that generated by the projections $M^{ij} = m^{ij}_{\mu\nu} M^{\mu\nu}$ and $M^{+-} = m^{+-}_{\mu\nu} M^{\mu\nu}$, which together generate the subgroup $SO(24) \times SO(1,1)$, with the transformation properties

$$[M^{+-}, Q^i] = Q^i \ \ ; \ \ [M^{+-}, Q^o] = Q^o \ , \tag{120}$$

and

$$[M^{ij}, Q^o] = 0 \ \ ; \ \ [M^{ij}, Q^k] = i\delta^{ik} Q^j - i\delta^{jk} Q^i \ . \tag{121}$$

In this very simple example, we see that the algebra preserves Lorentz invariance in $1+1$ dimensions, together with an internal $SO(24)$ algebra, and two nilpotent operators. Note that the set $\{Q^I\}$ forms a reducible representation of the unbroken algebra made up of a singlet Q^o and the Q^i which form a vector representation under the transverse subgroup.

Having the nilpotent operator Q^0, we may ask what nontrivial cohomology is possessed by the operator. For simplicity we will look at the ghost-free

sector. States $|\psi>$ satisfying $Q^0|\psi>=0$ in this sector are linear combinations of states of the form

$$|\psi>=\prod_{n=1}^{\infty}(a_\mu\alpha^\mu_{-n})^{p_n}|0> \tag{122}$$

where the α^μ_n are the Fourier modes of $X^\mu(\sigma)$ (not to be confused with the matrices $\alpha_{\mu\nu}$), and the ground state $|0>$ is annihilated by α_n and c_n for $n>0$ and b_n for $n\geq 0$. These states have zero norm, so they are trivial in the BRST sense, i.e. they can be written as Q acting on some other state. However not all of them can be written as Q^0 on other states. Specifically, states which can be so written are of the form

$$|\psi>_{triv}=\prod_{n=1}^{\infty}a\cdot\alpha_{-n}L^0_{-m}|0> \tag{123}$$

for any $m>0$, where

$$L^0_m\equiv\sum_p a\cdot\alpha_p a\cdot\alpha_{m-p}. \tag{124}$$

The set of states which are not of this form gives the cohomology of the nilpotent operator Q^0. Since these states are BRST trivial, they do not contribute to the dynamics of the usual BRST-based string theory. They would represent the physical states in some hypothetical theory based on Q^0.

Lorentz Symmetry Breaking

We have seen that only a subalgebra of the Lorentz algebra will transform the set $\{Q^I\}$ into itself; thus the set of Q^I's forms a representation (in general reducible) of the unbroken subalgebra. Put differently, under the relevant decomposition of the $SO(25,1)$ Lorentz algebra, $Q_{\mu\nu}$ itself decomposes into a sum of representations of the subalgebra, in terms of which the Q^I must be expressed. This is a powerful requirement which can be used to rule out some imbeddings. For instance, consider the imbedding of F_4 into the Lorentz

algebra. It does not work because the symmetric traceless tensor splits up into two F_4 representations, the smallest having 26 dimensions while there are only 25 Q^I's.

For any matrix α^I, the Lorentz-transformed quantity $\delta\alpha^I$ will automatically satisfy $\mathrm{Tr}\,\delta\alpha^I = 0$ and $\mathrm{Tr}\,\alpha^I\delta\alpha^I = 0$, because these quantities are the Lorentz transformations of the invariant traces $\mathrm{Tr}\,\alpha^I$ and $\mathrm{Tr}(\alpha^I)^2$ respectively. We can think of starting with some matrix α and building a set by applying the Lorentz transformations of some subgroup until a set closed under the subgroup is obtained. However at each step the condition $\mathrm{Tr}(\delta\alpha^I)^2 = 0$ must be checked for all δ in the unbroken subalgebra.

Carrying out this analysis (see appendix C), we find that the possibilities are:

1) The space of α's is 26-dimensional, but has no nilpotent element. There is no unbroken subgroup of the Lorentz group.

2) The space is 26-dimensional and contains a nilpotent element; the unbroken subgroup is $SO(1,1)$.

3) The space is d-dimensional, where $0 < d < 26$, with no nilpotent; the unbroken subgroup if $SO(26-d)$.

4) The space has dimension d, where $0 < d < 26$, and has a nilpotent element; the unbroken subgroup is $SO(1,1) \otimes SO(26-d)$.

5) The space is zero dimensional, thus containing only the BRST charge which was subtracted out as the trace; the full Lorentz group is unbroken.

The lack of a scheme in which the noncompact part of the unbroken subgroup is $SO(3,1)$ seems particularly unfortunate for any phenomenological applications of this procedure.

In conclusion, we have drawn attention to a curious algebra of normal-ordered, reparametrization invariant operators. The algebra closes only in a very few cases, one of which gives only the BRST charge.We have learned from this exercise that the requirement of reparametrization invariance in the oscillator basis can lead to the standard BRST results. We have arrived at this conclusion by demanding algebraic consistency. It is clear that in order to go further in this approach, we would have to find a different basis, i.e. a new ground state.

INTERACTIONS

We now study the important question of interactions in string theory. We will begin with a general discussion of the transition from quantum mechanics to field theory, in order to make several relevant points. We will then discuss reparametrization invariance in Witten's string field theory, the most well-studied example of an interacting string field theory. We will show that the apparent lack of reparametrization invariance is due to a parametrization-dependent choice of phases in a set of basis functions. We will construct an equivalent theory with an invariant choice of phases. This will result in an equivalent interaction which is explicitly reparametrization invariant, but not explicitly local in $x(\sigma)$.

Second Quantization

Let again consider the passage from the first quantized theory of open strings to the second quantized one. The Schrödinger wave functions in the first quantized theory are built as a linear combination of a set of basis functions:*

$$\Psi[x(\sigma)] = \sum_n c_n \; \psi_n[x(\sigma)] \tag{125}$$

The basis functions are in turn built by acting on the ground state with the creation operators α^{μ}_{-n}, where $n > 0$:

$$\psi_n[x] = (\alpha_{-1})^{n_1}(\alpha_{-2})^{n_2}... \; \psi_0[x] \tag{126}$$

* The string field of course is also a functional of the Faddeev-Popov ghosts. The treatment of the ghosts is in most cases identical to that of the coordinates; therefore we will not explicitly indicate such ghost dependence except in cases where the ghosts require special handling.

The ground state is annihilated by the destruction operators, which are α_n^μ for all positive n. The dynamics of the system are enforced by the Schrödinger equation, which in this case is

$$Q\Psi = 0 \tag{127}$$

where Q is the BRST operator.

In passage to the second-quantized theory, we define the (classical) string field $\Phi[x]$:

$$\Phi[x(\sigma)] = \sum_n a_n \ \psi_n[x(\sigma)] \tag{128}$$

This looks similar to (125). Indeed the string field (128) satisfies (127) as its classical equation of motion. However, there is an important distinction between (125) and (128): In (125) the coefficients c_n are complex numbers with arbitrary phases. Although the relative phases of two states will be important in a superposition such as (125), an individual phase by itself has no consequence. In the field theory defined by (128), the phases may be handled in two ways:

1) *Complex Field Theory:* The phases are allowed to take any value. The real and imaginary parts of the coefficients a_n in (128) are independent, and upon quantization they will become associated with two different types of quanta. In this type of field theory there will be "strings" and "anti-strings". These ideas do not agree with the well- known picture of string interactions in which there is only one type of string.

2) *Real Field Theory:* The phases are fixed to some specified values. Which values are used is of no physical consequence. Changing the phase convention would not change the theory in any fundamental way. Indeed, the free equation of motion and the inner product are independent of the phase choices; however, if an interaction is added its form might appear to be quite different. This type

of field theory has only one type of quanta; the strings are their own "anti-strings".

In Witten's theory of interacting open strings, the phase condition is that

$$\Phi^*[x(\sigma)] = \Phi[x(\pi - \sigma)], \tag{129}$$

which is clearly parametrization dependent. In terms of the basis functions (126), this condition means that the coefficient a in (128) should be real for states built from an odd number of odd modes, and pure imaginary otherwise. In other words, in Witten's theory, $i^{n+1}\alpha_n$ are "real" operators, in the sense that acting with them preserves the phase condition.

Witten's Interaction

Witten's interaction [24] is constructed as a generalization of the interactions of point particles in local field theory. We may think of point particle interactions as the creation of several particles from the vacuum at some instant of space-time, in such a way that energy-momentum is conserved. Thus one or more particles will have negative energy, and thus may be interpreted as *incoming* positive energy particles. The result is that the interaction is formed as a simple product of fields at a point.

Witten's generalization of this involves the definition of a product operation which maps two string fields Φ, Υ into a single field $\Phi * \Upsilon$. It is of the form

$$(\Phi * \Upsilon)[x] = \int DyDz\ K[x,y,z]\ \Phi[y]\ \Upsilon[z] \tag{130}$$

where the kernel $K[x,y,z]$ is given by

$$K[x,y,z] = \prod_{\sigma=0}^{\pi/2} \delta[x(\sigma)-y(\sigma)] \prod_{\sigma=0}^{\pi/2} \delta[y(\pi-\sigma)-z(\sigma)] \prod_{\sigma=\pi/2}^{\pi} \delta[x(\sigma)-z(\sigma)] \tag{131}$$

A crucial consistency check is that $\Phi * \Phi$ must satisfy the phase condition (129) for any Φ which does; this is easily seen to be true in this case. The interaction vertex which is then found using the inner product (32) is then guaranteed to be real for Φ satisfying (129).

$$\begin{aligned} V\{\Phi\} &= \int Dx\ \Phi^*[x]\ (\Phi * \Phi)[x] \\ &= \int DxDyDz\ \tilde{K}[x,y,z]\ \Phi[x]\ \Phi[y]\ \Phi[z] \end{aligned} \tag{132}$$

where

$$\tilde{K}[x,y,z] = \prod_{\sigma=0}^{\pi/2} \delta[x(\pi-\sigma) - y(\sigma)] \prod_{\sigma=0}^{\pi/2} \delta[y(\pi-\sigma) - z(\sigma)] \prod_{\sigma=0}^{\pi/2} \delta[z(\pi-\sigma) - x(\sigma)] \tag{133}$$

The redefined kernel $\tilde{K}$ is cyclically symmetric in x, y, z. It satisfies the following overlap equations for all $\sigma < \pi/2$:

$$\Big(x(\sigma) - z(\pi-\sigma)\Big)\tilde{K}[x,y,z] = 0 \tag{134a}$$

$$\left(\frac{\delta}{\delta x(\sigma)} + \frac{\delta}{\delta z(\pi-\sigma)}\right)\tilde{K}[x,y,z] = 0 \tag{134b}$$

and all cyclic permutations in (x,y,z). Note that the solution is possible because the operators in (134 a) and (134 b) commute. We may interpret these equations as the generalization of momentum conservation, i.e. the density of momentum is conserved on the overlapping strings. These overlap equations are an alternate starting point leading to the interaction form (133).

We are now in a position to understand what becomes of reparametrization invariance in this theory. When we act with a reparametrization it will in general not preserve the phase condition (129). Therefore the reparametrization must be accompanied by an appropriate transformation of the phases of the coefficients of the basis functionals. The generators of reparametrizations are made up of Virasoro operators L_n. In this theory, $i^{n+1}\alpha_n$ are real operators,

hence L_n is real for even n and imaginary for odd n. Thus we need to shift the phases by using the real operators $i^n L_n$ to generate reparametrizations. The full generators of reparametrizations become $i^n L_n - i^{-n} L_{-n} \equiv i^n K_n$ which are known [27] to be an exact kinematical symmetry of Witten's theory.

We therefore see that this theory does in fact possess reparametrization invariance, where the reparametrizations must be accompanied by corresponding phase shifts in the coefficients of the basis functionals. We could have chosen the basis functionals in a way such that the L_n operators are all real, in other words so that reparametrizations do not affect the phase choice. This will be the subject of the next section.

Manifestly Invariant Formulation

We may restore manifest parametrization invariance by choosing an invariant phase condition, namely

$$\Phi^*[x(\sigma)] = \Phi[x(\sigma)] \tag{135}$$

This condition makes all α_n pure imaginary operators, therefore all L_n will be real. We can translate the form of the overlap equations (134) to the new phase convention by the substitution

$$\alpha_n \to i^n \alpha_n \tag{136}$$

It is convenient to work in terms of the left and right combinations. Under (136) we find, using (62),

$$x_L(\sigma) \to \begin{cases} x_L(\sigma + \frac{\pi}{2}) + \frac{\pi}{2}\alpha_0, & \text{for } \sigma < \frac{\pi}{2}; \\ x_R(\frac{3\pi}{2} - \sigma) - \frac{3\pi}{2}\alpha_0, & \text{for } \sigma > \frac{\pi}{2}. \end{cases} \tag{137a}$$

$$x_R(\sigma) \to \begin{cases} x_L(\frac{\pi}{2} - \sigma) + \frac{\pi}{2}\alpha_0, & \text{for } \sigma < \frac{\pi}{2}; \\ x_R(\sigma - \frac{\pi}{2}) + \frac{\pi}{2}\alpha_0, & \text{for } \sigma > \frac{\pi}{2}. \end{cases} \tag{137b}$$

resulting in the new overlap equations, valid for $0 < \sigma < \pi$:

$$\Big(x_L(\sigma) + \frac{\pi}{2}\alpha_0^x - z_R(\sigma) - \frac{\pi}{2}\alpha_0^z\Big) K[x,y,z] = 0 \tag{138}$$

and all cyclic permutations in (x, y, z). Note that these equations are reparametrization invariant, since left and right combinations transform independently, and α_0 (the string's total momentum) is an invariant. The equations are consistent, since the commutator of x_L with itself cancels with that of z_R with itself. The solution to these equations (see Appendix D) does not involve delta function overlaps; rather it is*

$$\begin{aligned} K[x,y,z] = \delta[x(\pi) + x(0) - z(\pi) - z(0)]\ \delta[y(\pi) + y(0) - z(\pi) - z(0)] \\ \exp\left[-\frac{i}{2}\int \frac{d\sigma}{\pi}\Big(xy' + yz' + zx' - yx' - zy' - xz'\Big)\right] \end{aligned} \tag{139}$$

Several remarks about (139) need to be made. First, the overlap equations (138) do not hold at the endpoints $\sigma = 0, \pi$ due to surface terms in the integral over σ in (139). This is to be expected because the overlap equations at the endpoints came from those at the midpoint in the original Witten formulation, and $x'(\sigma)$ was discontinuous at the midpoint. Transforming the phases we see that at the endpoints only the following equations need be satisfied:

$$x(\pi) + x(0) = y(\pi) + y(0) = z(\pi) + z(0) \tag{140a}$$

$$\left(\frac{\delta}{\delta x(0)} + \frac{\delta}{\delta x(\pi)} + \frac{\delta}{\delta y(0)} + \frac{\delta}{\delta y(\pi)} + \frac{\delta}{\delta z(0)} + \frac{\delta}{\delta z(\pi)}\right) K[x,y,z] = 0 \tag{140b}$$

Hence the delta function overlaps at the endpoints. Note that the equations do not require e.g. $x(0) = z(\pi)$ as in the previous formulation.

Secondly, the solution gives a manifestly reparametrization invariant interaction vertex (up to anomalies, which will be discussed shortly). It is easy

* Because of (135), there is no distinction between K and $\tilde{K}$ as in the last section. Note also that no normal ordering is implied in the exponential; the K's are functionals, not operators which act on functionals.

to see this directly using the generator (35). Applying this to the interaction (132) we find, after integration by parts, the condition for an invariant vertex is

$$: \left(x'(\sigma) \cdot \frac{\delta}{\delta x(\sigma)} + y'(\sigma) \cdot \frac{\delta}{\delta y(\sigma)} + z'(\sigma) \cdot \frac{\delta}{\delta z(\sigma)} \right) : K[x,y,z] = 0 \qquad (141)$$

which can be easily shown to hold (disregarding for now the anomalous effect of the normal ordering) for (139). Alternatively, we note that (for example) $x(\sigma)$ and $y'(\sigma)$ transform covariantly with weights 0 and 1 respectively. Since there is no ordering ambiguity in the product xy', it has weight 0+1=1, thus its integral is an invariant. Note that the endpoint terms in (139) do not affect reparametrization invariance since the endpoints of the string are unaffected by reparametrizations (they are always fixed at $0, \pi$).

Thirdly, this vertex is guaranteed to possess all of the appealing properties of Witten's vertex, such as unitarity, correct modular properties, and BRST gauge invariance which persist despite ordering difficulties. Since our interaction is just Witten's with changes in the phase conventions, these properties follow automatically.

Finally, we need to check that (139) gives a vertex which gives a real number when evaluated for a function satisfying (135). This is indeed seen to be true for the particular form (139). Using (135),

$$(V\{\Phi\})^* = \int DxDyDz\ K^*[x,y,z]\ \Phi[x]\ \Phi[y]\ \Phi[z] \qquad (142)$$

but from (139),

$$K^*[x,y,z] = K[y,x,z] \qquad (143)$$

so, relabelling $x \leftrightarrow y$, we see that (142) is equal to (132).

Nonlocality

We note that the interactions in this formulation of the theory do not possess manifest spacetime locality; the interactions do not consist of delta function overlaps, except for the constraint that the position midway between the two endpoints must be the same for all three strings. In exposing the hidden reparametrization invariance of the theory we seem to have hidden the locality in the same way.

We should remember, however, that Witten's theory has a different kind of nonlocality, namely, nonlocality in terms of components. Whereas the action of a local theory contain only a finite number of derivatives, the vertices of any string field theory when written in components as (42,43) contain factors of $\exp\left(\frac{\partial}{\partial x_0}\right)^2$, which are responsible for the ultraviolet convergence of the theory. This can be traced to the fact that strings with different values of the zero mode coordinate x_0 may interact. Thus when the field theory is written in terms of components so that x_0 is the coordinate and the other modes x_n are internal degrees of freedom, the interactions are nonlocal. This nonlocality does not show up in perturbation theory, however it poses a great problem in any nonperturbative formulation of the theory [52] which has still not been solved. A discussion of this problem would be beyond the scope of this thesis; we simply remark that in terms of components, the new formulation is no less (and no more) local than in Witten's original formulation.

Ghost Insertion

Up to now we have not discussed the ghost part of the vertex. The same analysis applies in this case, and the result is almost identical. However, we know that in Witten's formulation there is an extra ghost insertion $\exp \frac{3}{2}\phi(\frac{\pi}{2})$,

which is necessary so that the interaction term in the action has the same ghost number as the kinetic term. Carrying out our substitution, we find that the required ghost insertion is $\exp \frac{3}{4}\Big(\phi(0) + \phi(\pi)\Big)$. This appears to be parametrization invariant because it is made from endpoint contributions; however, following the analysis of Gross and Jevicki [53], we find that this term gives an anomalous contribution to the effect of the reparametrization which exactly cancels the anomaly due to the normal ordering of the generators.

Finally, we give the complete form of the vertex, including the ghosts:

$$\begin{aligned} K[x,y,z;\phi,\xi,\chi] =& \delta[x(\pi)+x(0)-z(\pi)-z(0)]\ \delta[y(\pi)+y(0)-z(\pi)-z(0)] \\ & \delta[\phi(\pi)+\phi(0)-\chi(\pi)-\chi(0)]\ \delta[\xi(\pi)+\xi(0)-\chi(\pi)-\chi(0)] \\ & \exp\Big[-\frac{i}{2}\int\frac{d\sigma}{\pi}\Big(xy'+yz'+zx'-yx'-zy'-xz' \\ & +\phi\xi'+\xi\chi'+\chi\phi'-\xi\phi'-\chi\xi'-\phi\chi'\Big)\Big]\exp\frac{3}{4}\Big(\phi(\pi)+\phi(0)\Big) \end{aligned} \tag{144}$$

CONCLUSIONS

We have arrived, through our algebraic and group-theoretical approach, at our final result, which is an understanding of the apparent lack of reparametrization invariance in Witten's interacting string field theory. We have shown that this is due to the phase convention used in building the string field. We have shown one example of the same theory written using a different phase convention, choosing the particular choice which maintains reparametrization invariance. It is clear that one could, if one wished, construct the theory with any arbitrary phase conventions.

To conclude, we identify directions for future research in this subject matter:

(1) The field theory of closed strings must be developed. Although there has been progress in this area, it is clear that closed string field theory is not well understood. The interaction term between open and closed string fields must also be found.

(2) The issue of the nonlocality of string field theory must be dealt with. Some new formulation of the theory may be necessary, which is not manifestly perturbative, but one in which the nonlocality does not render the theory unsuitable. At present it is hard to imagine what this formulation might be.

(3) Some method for calculating in the low-energy limit of string physics must be developed and applied. Since exactly soluble models are rare, it is likely that some systematic approximation method will be appropriate. After such a procedure is developed, we will be able to properly examine string theory to decide whether it is or is not able to describe the universe that we live in.

Finally let us briefly mention what work is already under progress to address these issues:

(1) Some work still continues into the development of string field theories with closed strings. However the prevailing belief among string theorists is that string field theory is not the correct formalism through which to study string physics.

(2) Many ideas have been proposed which purport to be new nonperturbative formulations of string theory. So-called "matrix models" were speculated to be one such candidate recently. It is possible to do a few nonperturbative calculations in this formalism. However, as these models require the dimension of spacetime to be one or less, it seems unlikely that these models will be related in any way to reality. Related models such as topological field theories and W-gravity theories have been considered; while these models are of mathematical interest, they do not seem to describe the world in which we live. Yet another string-inspired approach is to consider models [54] which are nonlocal but which do not include the higher modes of the string. Such theories may evade the nonperturbative problems associated with nonlocality by the fact that they are local at the classical level. This approach can produce a finite theory of gravity, however since this form of nonlocalization is apparently possible for any field theory, and in fact there are many ways to do it for each theory, it does not reduce the arbitrariness of the standard model (in fact, it makes the problem worse).

(3) So far no reasonable justification has been suggested for the belief that strings will naturally compactify to give the standard model (or that they will compactify at all). In the formalism of string field theory it seems evident that no compactification will be stable, because of the nonlocality problem [52].

Other formalisms currently known depend on perturbation theory about a particular spacetime background so it is not possible to even ask such questions in these formalisms. The difficulty of these questions has led many researchers to abandon searching for realistic models and instead to study unrealistic theories which are solvable, in the hope that some insight may be gained. Others now believe that any attempt at studying particle physics at the inaccessibly high scales at which gravitational effects are significant is futile due to the lack of experimental evidence, and instead confine their efforts to physics at lower energy scales. In the absence of any great new ideas, it seems difficult to find a path between these two extreme points of view.

APPENDIX A

NOTATION AND CONVENTIONS

General

$$\text{Metric: } g_{00} = -1, \ g_{ii} = 1 \tag{A.1}$$

$$A'(\sigma) \equiv \frac{dA}{d\sigma} \tag{A.2}$$

For open strings, $0 < \sigma < \pi$.

Reparametrization representations

For a quantity $q(\sigma)$ transforming according to (w, z),

$$\delta_f q = -(f q' + w f' q + z f') \tag{A.3}$$

Examples:

String coordinates $x^\mu(\sigma)$ transform as $(0, 0)$.

Bosonized ghosts $\phi(\sigma)$ transform as $(0, -3/2)$.

Algebra:

$$[\delta_f, \delta_g] = \delta_{fg' - f'g} \tag{A.4}$$

Product rules: For p, q transforming as $(w_p, 0)$ and $(w_q, 0)$,

pq transforms as $(w_p + w_q, 0)$;

$w_p p q' - w_q p' q$ transforms as $(w_p + w_q + 1, 0)$;

Commutation Relations

$$[\frac{\delta}{\delta x^{\mu}(\sigma_1)}, x^{\nu}(\sigma_2)] = g^{\nu}_{\mu}\delta(\sigma_1 - \sigma_2) \tag{A.5a}$$

$$[\frac{\delta}{\delta \phi(\sigma_1)}, \phi(\sigma_2)] = \delta(\sigma_1 - \sigma_2) \tag{A.5b}$$

$$[\frac{\partial}{\partial x^{\mu}_m}, x^{\nu}_n] = g^{\nu}_{\mu}\delta_{m,n} \tag{A.6a}$$

$$[\frac{\partial}{\partial \phi_m}, \phi_n] = \delta_{m,n}. \tag{A.6b}$$

$$[x^{\mu}_L{}'(\sigma_1), x^{\nu}_L{}'(\sigma_2)] = -2i\pi g^{\mu\nu}\frac{d}{d\sigma_1}\delta(\sigma_1 - \sigma_2) \tag{A.7a}$$

$$[x^{\mu}_R{}'(\sigma_1), x^{\nu}_R{}'(\sigma_2)] = +2i\pi g^{\mu\nu}\frac{d}{d\sigma_1}\delta(\sigma_1 - \sigma_2) \tag{A.7b}$$

$$[\phi'_L(\sigma_1), \phi'_L(\sigma_2)] = +2i\pi\frac{d}{d\sigma_1}\delta(\sigma_1 - \sigma_2) \tag{A.7c}$$

$$[\phi'_R(\sigma_1), \phi'_R(\sigma_2)] = -2i\pi\frac{d}{d\sigma_1}\delta(\sigma_1 - \sigma_2) \tag{A.7d}$$

Mode Expansions

$$x^{\mu}(\sigma) = x_0 + \sqrt{2}\sum_{n=1}^{\infty} x_n \cos n\sigma \tag{A.8a}$$

$$\phi(\sigma) = \phi_0 + \sqrt{2}\sum_{n=1}^{\infty} \phi_n \cos n\sigma \tag{A.8b}$$

$$\frac{\delta}{\delta x^{\mu}(\sigma)} = \frac{1}{\pi}\left(\frac{\partial}{\partial x^{\mu}_0} + \sqrt{2}\sum_{n=1}^{\infty}\frac{\partial}{\partial x^{\mu}_n}\cos n\sigma\right) \tag{A.9a}$$

$$\frac{\delta}{\delta \phi(\sigma)} = \frac{1}{\pi}\left(\frac{\partial}{\partial \phi_0} + \sqrt{2}\sum_{n=1}^{\infty}\frac{\partial}{\partial \phi_n}\cos n\sigma\right) \tag{A.9b}$$

$$x^{\mu}_{L,R}{}'(\sigma) = x^{\mu\prime}(\sigma) \pm i\pi\frac{\delta}{\delta x_{\mu}(\sigma)} \tag{A.10a}$$

$$\phi'_{L,R}(\sigma) = \phi'(\sigma) \mp i\pi\frac{\delta}{\delta \phi(\sigma)} \tag{A.10b}$$

$$x^{\mu}_{L,R}(\sigma) = x^{\mu}_0 \mp \alpha^{\mu}_0 \sigma + i \sum_{n\neq 0} \frac{1}{n} \alpha^{\mu}_n e^{\mp in\sigma} \tag{A.11a}$$

$$\phi_{L,R}(\sigma) = \phi^{\mu}_0 \mp \beta^{\mu}_0 \sigma + i \sum_{n\neq 0} \frac{1}{n} \beta^{\mu}_n e^{\mp in\sigma} \tag{A.11b}$$

$$\alpha^{\mu}_{\pm n} = -i\sqrt{2}\left(\frac{\partial}{\partial x_{n\mu}} \pm n x^{\mu}_n\right) \tag{A.12a}$$

$$\beta_{\pm n} = +i\sqrt{2}\left(\frac{\partial}{\partial \phi_n} \pm n \phi_n\right) \tag{A.12b}$$

for $n > 0$, and

$$\alpha^{\mu}_0 = -i\frac{\partial}{\partial x_{0\mu}} \tag{A.12c}$$

$$\beta_0 = +i\frac{\partial}{\partial \phi_0} \tag{A.12d}$$

Generators

For any functional Φ,

$$\delta_f \Phi = -i\epsilon M_f \Phi \tag{A.13}$$

where

$$\begin{aligned} M_f^{Classical} \equiv -\, i\int_0^{\pi} d\sigma\, f(\sigma)\Big[& x'(\sigma)\cdot\frac{\delta}{\delta x(\sigma)} \\ & +\phi'(\sigma)\frac{\delta}{\delta\phi(\sigma)} + \frac{3}{2}\left(\frac{\delta}{\delta\phi(\sigma)}\right)'\Big] \end{aligned} \tag{A.14}$$

$$[M_f, M_g] = iM_{fg'-f'g} \tag{A.15}$$

$$M_f =: M_f^{Classical} : \tag{A.16}$$

$$M_{\sqrt{2}\sin n\sigma} = -\frac{i}{\sqrt{2}}(L_n - L_{-n}) \tag{A.17}$$

$$M^{Classical}_{\sqrt{2}\sin n\sigma} = -\frac{i}{\sqrt{2}}(L_n - L_{-n} + C_n) \tag{A.18}$$

$$C_n = \begin{cases} \frac{n(D+1)}{8} & \text{for } n \text{ even,} \\ 0 & \text{for } n \text{ odd.} \end{cases} \tag{A.19}$$

$$L_n = L^x_n + L^{\phi}_n \tag{A.20}$$

$$L_n^x \equiv \frac{1}{2}\sum_l \alpha_l \cdot \alpha_{n-l} \tag{A.21a}$$

$$L_n^\phi \equiv \frac{1}{2}\sum_l \beta_l \beta_{n-l} + i\frac{wn}{2}\beta_n \tag{A.21b}$$

$$M_f^L = -\frac{1}{4}\int_0^\pi \frac{d\sigma}{\pi} f(\sigma)\left({x'_L}^2 - {\phi'_L}^2 - 3\phi''_L\right) \tag{A.22a}$$

$$M_f^R = +\frac{1}{4}\int_0^\pi \frac{d\sigma}{\pi} f(\sigma)\left({x'_R}^2 - {\phi'_R}^2 - 3\phi''_R\right) \tag{A.22b}$$

Invariant Operators

$$p_\mu \equiv -i\int_0^\pi d\sigma \frac{\delta}{\delta x^\mu(\sigma)} \tag{A.23}$$

$$N_G \equiv -i\int_0^\pi d\sigma \frac{\delta}{\delta \phi(\sigma)} \tag{A.24}$$

$$M_{\mu\nu} \equiv i\int_0^\pi d\sigma \left(x_\mu \frac{\delta}{\delta x^\nu} - x_\nu \frac{\delta}{\delta x^\mu}\right) \tag{A.25}$$

$$Q^{\mu\nu} \equiv \int_0^\pi \frac{d\sigma}{\pi} : e^{\phi_L}\left[x_L'^\mu x_L'^\nu - \frac{g^{\mu\nu}}{26}(\phi_L'^2 + 3\phi_L'')\right] : +(L \to R). \tag{A.26}$$

$$Q = g_{\mu\nu}Q^{\mu\nu} \tag{A.27}$$

$$B^{\mu\nu} = \int_0^\pi \frac{d\sigma}{\pi} : e^{2\phi_L}(x_L'^\mu x_L'^\nu - \frac{g^{\mu\nu}}{26}x'_L \cdot x'_L) + (L \to R) : \tag{A.28}$$

String Field Dynamics

Equation of Motion: $\qquad Q\Phi + \Phi * \Phi = 0$

Action: $\qquad S\{\Phi\} = \frac{1}{2} < \Phi|Q|\Phi > + \frac{1}{3}V\{\Phi\}$

$$< \Phi|\Upsilon >= \int Dx \Phi^*[x]\Upsilon[x] \tag{A.29}$$

$$(\Phi * \Upsilon)[x] = \int DyDz\ K[x,y,z]\ \Phi[y]\ \Upsilon[z] \tag{A.30}$$

$$
\begin{aligned}
V\{\Phi\} &= \int Dx\ \Phi^*[x]\ (\Phi * \Phi)[x] \\
&= \int DxDyDz\ \tilde{K}[x,y,z]\ \Phi[x]\ \Phi[y]\ \Phi[z]
\end{aligned}
\tag{A.31}
$$

Witten Formulation:

$$
\Phi^*[x(\sigma)] = \Phi[x(\pi - \sigma)] \tag{A.32}
$$

$$
K[x,y,z] = \prod_{\sigma=0}^{\pi/2} \delta[x(\sigma)-y(\sigma)] \prod_{\sigma=0}^{\pi/2} \delta[y(\pi-\sigma)-z(\sigma)] \prod_{\sigma=\pi/2}^{\pi} \delta[x(\sigma)-z(\sigma)] \tag{A.33}
$$

$$
\tilde{K}[x,y,z] = \prod_{\sigma=0}^{\pi/2} \delta[x(\pi-\sigma)-y(\sigma)] \prod_{\sigma=0}^{\pi/2} \delta[y(\pi-\sigma)-z(\sigma)] \prod_{\sigma=0}^{\pi/2} \delta[z(\pi-\sigma)-x(\sigma)] \tag{A.34}
$$

New Formulation:

$$
\Phi^*[x] = \Phi[x] \tag{A.35}
$$

$$
\begin{aligned}
K[x,y,z] &= \tilde{K}[x,y,z] \\
&= \delta[x(\pi)+x(0)-z(\pi)-z(0)]\ \delta[y(\pi)+y(0)-z(\pi)-z(0)] \\
&\quad \exp\left[-\frac{i}{2}\int \frac{d\sigma}{\pi}\Big(xy' + yz' + zx' - yx' - zy' - xz'\Big)\right]
\end{aligned}
\tag{A.36}
$$

APPENDIX B

UNIQUENESS OF REPRESENTATIONS

In this appendix we shall prove that the doublet representation given by (17) and (23) is the only irreducible linear representation of the super-reparametrization algebra whose basis elements are a finite number of fields which transform covariantly under reparametrizations. We will show that given a set of covariant fields which transform into each other under super-reparametrizations, the representation can be reduced into a series of doublets. We will use the notation $a_{w,i}$ to denote the ith field of weight w in the collection, where $i = 1$ to N_w for each value of w.

Consider the fields $a_{w_0,i}$, where w_0 is the lowest weight in the set. Since the super-reparametrizations increase the weight by $\frac{1}{2}$, these fields must transform into weight $w_0 + \frac{1}{2}$ fields. We can choose the basis for these fields so that

$$\not{\delta}_f a_{w_0,i} = -f a_{w_0+\frac{1}{2},i}, \qquad i = 1 \text{ to } N_{w_0} \tag{B.1}$$

Applying a second super-reparametrization operator, the covariance of $a_{w_0,i}$ requires

$$\not{\delta}_f a_{w_0+\frac{1}{2},i} = -(f a'_{w_0,i} + 2w_0 f' a_{w_0,i}) \qquad i = 1 \text{ to } N_{w_0} \tag{B.2}$$

i.e. the combinations $(a_{w_0,i}, a_{w_0+\frac{1}{2},i})$ form N_{w_0} independent doublets. We now show that with an appropriate choice of basis, the elements of these doublets do not appear elsewhere in the representation. First consider the other elements,

i.e. $N_{w_0} < i \le N_{w_0+\frac{1}{2}}$. The most general possible transformation law satisfying (16) for these elements is

$$\delta\!\!\!/_f a_{w_0+\frac{1}{2},i} = -\sum_{j=1}^{N_{w_0}} A_{ij}(f a'_{w_0,j} + 2w_0 f' a_{w_0,j}) - f a_{w_0+1,i} \tag{B.3}$$

with an appropriate choice of basis for the weight w_0+1 elements. By changing the basis for the weight $w_0 + \frac{1}{2}$ elements we can obtain elements which do not transform into the weight w_0 elements. Redefining

$$a_{w_0+\frac{1}{2},i} \to a_{w_0+\frac{1}{2},i} - \sum_{j=1}^{N_{w_0}} A_{ij} a_{w_0+\frac{1}{2},j}, \tag{B.4}$$

we obtain

$$\delta\!\!\!/ a_{w_0+\frac{1}{2},i} = -f a_{w_0+1,i} \tag{B.5}$$

We now show that the elements in the doublets $(a_{w_0}, a_{w_0+\frac{1}{2}})$ do not appear elsewhere in the algebra. (Here the subscripts i are left as implicitly understood). Let a_w be the first (i.e. lowest weight) element whose transformation law involves one of these elements. Then there are two cases to be considered:

1) $w - w_0 \equiv n$ is an integer, and the transformation of a_w involves $a_{w_0+\frac{1}{2}}$. a_w could possibly have the transformation law

$$\delta\!\!\!/_f a_w = \sum_m A_m \frac{d^m f}{d\sigma^m} \frac{d^{n-m} a_{w_0+\frac{1}{2}}}{d\sigma^{n-m}} + X_f \tag{B.6}$$

where A_i are coefficients and X_f is some quantity which does not involve the elements in the doublet. X_f is found to transform to

$$\delta\!\!\!/_f X_f = f^2 a'_w + 2w f f' a_w + \sum_m A_m \frac{d^m f}{d\sigma^m} \frac{d^{n-m}}{d\sigma^{n-m}} (f a'_{w_0} + 2w_0 f' a_{w_0}). \tag{B.7}$$

The transformation of X_f involves a_{w_0}; since we assumed that no field of lower weight than w has this property, X_f must have higher weight; the only possibility is

$$X_f = f a_{w+\frac{1}{2}}. \tag{B.8}$$

Since X_f has no derivatives of f, the only possible A_i's which could be nonzero are those which are multiplied by f, which in this case is only A_0. Then we find

$$\begin{aligned}\oint_f^2 a_{w+\frac{1}{2}} =& A_0 \frac{d}{d\sigma}\left(f(-f a_{w_0+\frac{1}{2}})' + 2w_0 f'(-f a_{w_0+\frac{1}{2}})) + 2w_0 f'(-f a_{w_0+\frac{1}{2}})\right) \\ &+ f(\oint_f a_w)' + 2w \oint_f a_w\end{aligned} \tag{B.9}$$

It is easy to see by substituting from (B.6) that this cannot be satisfied unless A_0=0.

2) $w - w_0$ is half integral ($w = w_0 + n - \frac{1}{2}$), and the transformation of a_w involves a_{w_0}. The details of this case are similar to case 1. We find

$$\oint_f a_w = \sum_{m=1}^{n} A_m \frac{d^m f}{d\sigma^m} \frac{d^{n-m}}{d\sigma^{n-m}} a_{w_0} + f a_{w+\frac{1}{2}} \tag{B.10}$$

and

$$f \oint_f a_{w+\frac{1}{2}} = \sum_m A_m \frac{d^m f}{d\sigma^m} \frac{d^{n-m}}{d\sigma^{n-m}} \left(f a_{w_0+\frac{1}{2}}\right) + f^2 a_w' + 2w f f' a_w \tag{B.11}$$

requiring $A_m = 0$ except for $m = 0$ and $m = n$; then

$$\begin{aligned}\oint_f^2 a_{w+\frac{1}{2}} =& A_0 \frac{d^n}{d\sigma^n}\left(-f^2 a_{w_0}' - 2w_0 f f' a_{w_0}\right) + A_n \frac{d^n f}{d\sigma^n}\left(-f a_{w_0}' - 2w_0 f' a_{w_0}\right) \\ &+ f \frac{d}{d\sigma}\left(A_0 \frac{d^n}{d\sigma^n}(f a_{w_0+\frac{1}{2}}) + A_n \frac{d^n f}{d\sigma^n} a_{w_0+\frac{1}{2}}\right) \\ &+ 2w f' \left(A_0 \frac{d^n}{d\sigma^n}(f a_{w_0+\frac{1}{2}}) + A_n \frac{d^n f}{d\sigma^n} a_{w_0+\frac{1}{2}}\right)\end{aligned} \tag{B.12}$$

which again cannot be satisfied unless A_0 and A_n are zero.

We have shown that the lowest weight fields are part of doublets which decouple from all other fields under super-reparametrizations. One may apply the same procedure to what remains, again and again until the whole representation is reduced to doublets. So any arbitrary representation in terms of covariant quantities may be reduced to doublets.

APPENDIX C

UNBROKEN LORENTZ SUBGROUPS

In this appendix we derive the results of chapter 4, namely the different possible subgroups of the Lorentz group which may be unbroken by the set of matrices $\alpha_{\mu\nu}$.

Under a Lorentz transformation $\delta_{\alpha\beta}$, the matrix $\alpha_{\mu\nu}$ transforms as

$$\delta_{\alpha\beta}\alpha_{\mu\nu} = g_{\alpha\mu}\alpha_{\beta\nu} - g_{\beta\mu}\alpha_{\alpha\nu} + g_{\alpha\nu}\alpha_{\mu\beta} - g_{\beta\nu}\alpha_{\mu\alpha} \tag{C.1}$$

We see that the Lorentz-transformed quantity $\delta\alpha$ will automatically satisfy $\operatorname{Tr}\delta\alpha = 0$ and $\operatorname{Tr}\alpha\delta\alpha = 0$, because these quantities are the Lorentz transformations of the invariant traces $\operatorname{Tr}\alpha$ and $\operatorname{Tr}(\alpha^2)$ respectively. We can think of starting with some matrix α and building a set by applying the Lorentz transformations of some subgroup until a set closed under the subgroup is obtained. However at each step the condition $\operatorname{Tr}(\delta\alpha)^2 = 0$ must be checked for all δ in the unbroken subalgebra.

We have already shown an example which preserves $SO(1,1)$ symmetry. Let us postulate an $SO(2,1)$ unbroken symmetry, and show that a contradiction results. By a change of basis we can arrange things so that the hypothetical unbroken symmetry acts on the first three coordinates, so the unbroken generators are δ_{01}, δ_{02}, and δ_{12}.

We can take a set of α's which transforms irreducibly under the unbroken symmetry, i.e. one which is zero except in the first 3 by 3 block. To prove that this is always possible, we notice that acting on any general α with any of the unbroken generators produces another α (which, by the assumption of

unbroken symmetry, must still be in the set) which has zeros in the last 23 by 23 block. Further application of these transformation will not yield any nonzero contribution to this block. Therefore there exists a subset of the α matrices, with the same unbroken symmetry, whose last 23 by 23 blocks are zero. Now we can obtain α matrices whose 3 by 23 off-diagonal blocks are zero, by acting with the casmir operator $2(\delta_{01}^2 + \delta_{02}^2 + \delta_{12}^2) - 1$, which will leave the first 3 by 3 block unchanged, but annihilate the off-diagonal blocks.

Thus without loss of generality we may concentrate on a set of matrices of the form

$$\alpha_{\mu\nu} = \begin{pmatrix} A & B & C \\ B & D & E \\ C & E & F \end{pmatrix} \tag{C.2}$$

where we have shown only the first three rows and columns. The trace conditions on this matrix require

$$A = D + F \tag{C.3a}$$

and

$$D^2 + E^2 + F^2 + DF = B^2 + C^2 \tag{C.3b}$$

Computing the transformed matrices,

$$\delta_{01}\alpha = 2\begin{pmatrix} 2B & A+D & E \\ A+D & 2B & C \\ E & C & 0 \end{pmatrix} \tag{C.4a}$$

$$\delta_{02}\alpha = 2\begin{pmatrix} 2C & E & A+F \\ E & 0 & B \\ A+F & B & 2C \end{pmatrix} \tag{C.4b}$$

$$\delta_{01}\alpha = 2\begin{pmatrix} 0 & C & -B \\ C & 2E & F-D \\ -B & F-D & -2E \end{pmatrix} \tag{C.4c}$$

which (as advertised) are automatically traceless; however, we must impose the condition that the trace of the squared matrix is zero. This gives three equations:

$$4C^2 + B^2 = E^2 + (A+D)^2 \tag{C.5a}$$

$$B^2 + 4C^2 = E^2 + (A+F)^2 \tag{C.5b}$$

$$4E^2 + (F-D)^2 = C^2 + B^2 \tag{C.5c}$$

Solving (C.3) and (C.5), we find the general solution

$$\alpha = \begin{pmatrix} D+F & \sqrt{D^2+DF} & \sqrt{F^2+DF} \\ \sqrt{D^2+DF} & D & \sqrt{DF} \\ \sqrt{F^2+DF} & \sqrt{DF} & F \end{pmatrix} \tag{C.6}$$

which has the form $a_\mu a_\nu$, where $a_\mu = (\sqrt{D+F}, \sqrt{D}, \sqrt{F})$. This is the same form as the nilpotent matrix in the example of chapter 4. However, we have shown that there can be only one nilpotent matrix in the set. Therefore the transformed matrices (C.4) cannot have this form, and indeed by examining them it is easy to see that they do not; therefore transforming a second time will necessarily take us out of the set. Thus we have shown that it is impossible to have an unbroken $SO(2,1)$ subgroup except in the trivial case when only the BRST charge is included. Higher subgroups $SO(N,1)$ are of course also excluded, since they contain $SO(2,1)$.

It is possible to have an unbroken $SO(n)$ subgroup, by using a smaller set of matrices. For example, if we use 23 matrices, which are all zero in the last three rows and columns, then we have an unbroken $SO(3)$ group. Such a construction does not work for the noncompact subgroups because taking the first row (the timelike one) and column to be zero makes it impossible to satisfy the trace conditions.

APPENDIX D

SOLUTION OF OVERLAP EQUATIONS

We present the details of the derivation of the solution (139) to the overlap equations (138). First we act with $\frac{d}{d\sigma}$ and reexpress the left and right combinations in terms of coordinates and functional derivatives; then the equations become

$$\left(x' + i\pi\frac{\delta}{\delta x} - z' + i\pi\frac{\delta}{\delta z}\right) K[x,y,z] = 0 \tag{D.1a}$$

$$\left(y' + i\pi\frac{\delta}{\delta y} - x' + i\pi\frac{\delta}{\delta x}\right) K[x,y,z] = 0 \tag{D.1b}$$

$$\left(z' + i\pi\frac{\delta}{\delta z} - y' + i\pi\frac{\delta}{\delta y}\right) K[x,y,z] = 0 \tag{D.1c}$$

We substitute $K = e^A$, and change variables to the quantities u, v, v^* which transform irreducibly under permutations of x, y, z:

$$u = x + y + z \tag{D.2a}$$

$$v = x + ey + e^*z \tag{D.2b}$$

$$v^* = x + e^*y + ez \tag{D.2c}$$

where $e \equiv \exp\frac{2i\pi}{3}$. The equations (D.1) become

$$6i\pi\frac{\delta}{\delta u}A = 0 \tag{D.3a}$$

$$(1-e)v' = i\pi(1+e)\frac{\delta}{\delta v^*}A \tag{D.3b}$$

$$(1-e^*)v^{*\prime} = i\pi(1+e^*)\frac{\delta}{\delta v}A \tag{D.3c}$$

Then using

$$\frac{1-e}{1+e} = -i\tan\frac{\pi}{3} = -i\sqrt{3} \tag{D.4}$$

we find the solution

$$A = -\frac{1}{2\sqrt{3}\pi} \int d\sigma (v^* v' - v^{*\prime} v) \tag{D.5}$$

where we have integrated by parts to write the solution in the form which will give a real vertex for fields satisfying the reality condition. Note that integration by parts will create surface terms which may affect the overlap equations at the endpoints, which will need to be checked subsequently. In terms of our original variables,

$$A = \frac{i}{2\pi}(x'y - xy' + y'z - yz' + xz' - x'z) \tag{D.6}$$

which is the solution (139). The endpoint overlaps require $x(0) + x(\pi)$ to be the same for all three strings. It is not hard to see that (D.6) is consistent with this, due to the total antisymmetry of A under permutations of x, y, z.

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I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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